Nonlinear Analysis: Real World Applications 13 (2012) 1955–1960

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Solutions for a fractional diffusion equation with noninteger dimensions

L.S. Lucena [a], L.R. da Silva [a], A.A. Tateishi [b], M.K. Lenzi [c], H.V. Ribeiro [b], E.K. Lenzi [b,*]

[a] Departamento de Física, Universidade Federal do Rio Grande do Norte, 59072 – 970, Natal, RN, Brazil

[b] Departamento de Física, Universidade Estadual de Maringá, Avenida Colombo 5790, 87020 – 900, Maringá, PR, Brazil

[c] Departamento de Engenharia Química, Universidade Federal do Paraná, Setor de Tecnologia – Jardim das Américas, Caixa Postal 19011, 81531 – 990, Curitiba, PR, Brazil

ARTICLE INFO

Article history:
Received 1 December 2011
Accepted 27 December 2011

Keywords:
Fractional diffusion equation
Anomalous diffusion
Noninteger dimension

ABSTRACT

We investigate a fractional diffusion equation with a nonlocal reaction term by using the Green function approach. We also consider a modified spatial operator in order to cover situations characterized by a noninteger dimension. The results show a nonusual spreading of the initial condition which can be connected to a rich class of anomalous diffusive processes.

1. Introduction

The non-Markovian characteristics of the processes related to anomalous diffusion [1–3] and its ubiquity in nature have motivated researchers of many scientific fields to investigate this subject. In fact, nonusual processes are present in several situations such as diffusion on fractals [4,5], biological cells [6,7], animal foraging behavior [8], *human epidemic dynamic model* [9], nanoscience [10,11], geophysical systems [12,13], motion of colloidal particles [14], systems with long range interactions [15,16], and the adsorption–desorption process [17–19]. One of the main characteristics of the anomalous diffusion is the time dependence of the mean square displacement which, depending on the stochastic process, may be expressed, for example, as $\langle (r-\langle r\rangle)^2\rangle \sim t^{\alpha}$, where $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$ correspond to sub-, normal and superdiffusive cases, respectively. It is also possible to have situations in which the mean square displacement is not defined [1]. This feature occurs when Lévy distributions are involved. *In order to cover these scenarios, some formalisms such as the random walks* [20], *master equations* [21], *Langevin equations* [2,22] *and diffusion equations* [23,24] *have been used with suitable considerations. In particular*, the diffusion equation has been extended by incorporating spatial and time fractional derivatives. A direct consequence is a nonusual spreading of the system which may be related to rich classes of anomalous diffusion. They have been systematically investigated such as the developments performed in Refs. [25–36]. In this sense, the Schrödinger equation has also been extended as shown in Refs. [37–43] by incorporating fractional derivatives, *α-dimensional fractional space* [44,45], and nonlocal terms [46,47]. *Furthermore, a Lippmann–Schwinger equation for the fractional quantum mechanics has been proposed in* Ref. [48]. In the framework of these formal developments, here the solutions of the following diffusion equation will be considered:

$$\frac{\partial^{\gamma}}{\partial t^{\gamma}}\rho(\mathbf{r},t)=\mathcal{D}\widetilde{\nabla}^{2}\rho(\mathbf{r},t)-\int_{0}^{t}d\bar{t}\,\alpha(t-\bar{t})\rho(\mathbf{r},\bar{t}),\tag{1}$$

* Corresponding author.
E-mail address: eklenzi@dfi.uem.br (E.K. Lenzi).

1468-1218/$ – see front matter
doi:10.1016/j.nonrwa.2011.12.025

where $\mathbf{r} = (r, \theta)$, $\mathcal{D}$ has the dimension $[L]^2/[T]^\gamma$ and for $\gamma = 1$, it recovers the usual dimension. The fractional time derivative considered in Eq. (1) is the Caputo one [49] and the spatial operator is given by

$$\widetilde{\nabla}^2\rho(\mathbf{r},t) \equiv \frac{1}{r^{\mathcal{N}_f-1}}\frac{\partial}{\partial r}\left(r^{\mathcal{N}_f-1}\frac{\partial}{\partial r}\rho(\mathbf{r},t)\right) + \frac{1}{r^2\sin^{\mathcal{N}_f-2}\theta}\frac{\partial}{\partial\theta}\left(\sin^{\mathcal{N}_f-2}\theta\frac{\partial}{\partial\theta}\rho(\mathbf{r},t)\right), \tag{2}$$

where $\mathcal{N}_f$ represents a noninteger dimension [44]. *It is interesting to note that the presence of the noninteger dimension has also been considered to investigate the electromagnetic wave propagation in fractional space* [50]. The propagator for Eq. (1) is first investigated by considering the system subjected to the boundary conditions $\rho(\mathbf{r},t)|_{r=a} = \Phi(\theta,t)$ and $\partial_r\rho(\mathbf{r},t)|_{r=0} = 0$. Next, we extend our analysis to the case $\lim_{|\mathbf{r}|\to\infty}\rho(\mathbf{r},t) = 0$. In both cases, we first obtain the solutions in the absence of the nonlocal term and later, we analyze the solutions by incorporating the nonlocal term where the kernel $\alpha(t)$ quantifies how the quantity $\rho(\mathbf{r},t)$ is influenced by $\rho(\mathbf{r},\bar{t})$. At this point, it is interesting to note that Eq. (1) has a particular case where the situations work out in Refs. [51–53]. These developments are performed in Section 2, while in Section 3, the discussions and conclusions are presented.

2. Fractional equation

Let us start our discussion by considering the previous diffusion equation in the absence of the nonlocal term and subjected to the boundary conditions $\partial_r\rho(\mathbf{r},t)|_{r=0} = 0$ and $\rho(\mathbf{r},t)|_{r=a} = \Phi(\theta,t)$. After that, we incorporate the nonlocal term and extend the result to the case $\lim_{r\to\infty}\rho(r,t) = 0$. For these cases, the initial condition is an arbitrary normalized distribution function in the interval considered. The diffusion equation to be considered is given by

$$\frac{\partial^\gamma}{\partial t^\gamma}\rho(\mathbf{r},t) = \mathcal{D}\widetilde{\nabla}^2\rho(\mathbf{r},t). \tag{3}$$

This equation has the usual diffusion equation for $\gamma = 1$ and $\mathcal{N}_f$ integer as a particular case. In order to solve the equation accomplishing the required boundary condition, we use the Laplace transform and the Green function approach. By performing the Laplace transform, we obtain

$$\mathcal{D}s^{1-\gamma}\widetilde{\nabla}^2\rho(\mathbf{r},t) - s\rho(\mathbf{r},s) = -\rho(\mathbf{r},0) \tag{4}$$

where the last term on the right side represents an arbitrary normalized initial condition. The formal solution for this equation is given by

$$\rho(\mathbf{r}',s) = -\int_0^a drr^{\mathcal{N}_f-1}\int_0^\pi d\theta\sin^{\mathcal{N}_f-2}\theta\rho(\mathbf{r}',0)\mathcal{G}(\mathbf{r},\mathbf{r}';s) - a^{\mathcal{N}_f-1}\int_0^\pi d\theta\sin^{\mathcal{N}_f-2}\theta\Phi(\theta,s)\frac{\partial}{\partial r}\mathcal{G}(\mathbf{r},\mathbf{r}';s)\bigg|_{r=a}, \tag{5}$$

with the propagator (or the Green function) determined by the equation

$$\mathcal{D}s^{1-\gamma}\widetilde{\nabla}^2\mathcal{G}(\mathbf{r},\mathbf{r}';s) - s\mathcal{G}(\mathbf{r},\mathbf{r}';s) = \frac{1}{r^{\mathcal{N}_f-1}\sin^{\mathcal{N}_f-2}\theta}\delta(r-r')\delta(\theta-\theta') \tag{6}$$

and subjected to the conditions $\partial_r\mathcal{G}(\mathbf{r},\mathbf{r}';s)|_{r=0} = 0$ and $\mathcal{G}(\mathbf{r},\mathbf{r}';s)|_{r=a} = 0$. Note that the last term of Eq. (5) represents the surface effects (e.g., adsorption–desorption processes) on the relaxation of the system's initial condition.

In order to obtain the solution for Eq. (6), we may use the eigenfunctions of the Sturm–Liouville problem related to the spatial operator, i.e., one solves the equation $\widetilde{\nabla}^2\psi = -k^2\psi$, with ψ satisfying the boundary conditions mentioned above for $\mathcal{G}(\mathbf{r},\mathbf{r}';s)$. In this sense, it is possible to show that

$$\mathcal{G}(\mathbf{r},\mathbf{r}';s) = \sum_{n=1,l=0}^{\infty}\mathcal{A}_{n,l}(\mathbf{r}',s)\psi(\mathbf{r},k_{n,l}) \tag{7}$$

with the eigenfunction given by

$$\psi(\mathbf{r},k_{n,l}) = \mathcal{N}_{n,l}r^{1-\alpha/2}J_\nu\left(k_{n,l}r\right)C_l^{\alpha/2-1}\left(\cos\theta\right) \tag{8}$$

where $J_\nu\left(k_{n,l}r\right)$ is the Bessel function with $\nu = \alpha/2 + l - 1$, $C_l^{\widetilde{\alpha}}\left(\cos\theta\right)$ are the Gegenbauer polynomials [54] with

$$\mathcal{A}_{n,l}(\mathbf{r}',s) = \frac{\psi(\mathbf{r}',k_{n,l})}{s + \mathcal{D}s^{1-\gamma}k_{n,l}^2}, \tag{9}$$

$$\mathcal{N}_{n,l}^2 = \frac{2}{a^2J_{\nu+1}^2\left(k_{n,l}a\right)}\frac{l!\left(l+\frac{\mathcal{N}_f}{2}-1\right)}{2^{3-\mathcal{N}_f}\pi\Gamma\left(l+\mathcal{N}_f-2\right)}\left[\Gamma\left(\frac{\mathcal{N}_f}{2}-1\right)\right]^2, \tag{10}$$

and the eigenvalues are determined by the equation $J_\nu\left(k_{n,l}a\right) = 0$. *Note that the solutions for* Eq. (6) *may also be found by using the procedure presented in* Ref. [55] *based on the finite Hankel transforms*. By performing the inverse Laplace transform, we obtain that

$$\mathcal{G}(\mathbf{r}, \mathbf{r}'; t) = \sum_{n=1,l=0}^{\infty} \psi(\mathbf{r}', k_{n,l})\psi(\mathbf{r}, k_{n,l})\mathrm{E}_\gamma\left(-\mathcal{D}k_{n,l}^2 t^\gamma\right). \tag{11}$$

The presence of the Mittag-Leffler function [49] in the Green function indicates an anomalous relaxation of the system which, for lesser times, is governed by a stretched exponential, i.e., $\mathrm{E}_\gamma(x) \sim e^{-x^\gamma/\Gamma(1+\gamma)}$, and, for longer times, a power-law behavior, i.e., $\mathrm{E}_\gamma(x) \sim -1/\left(\Gamma(1-\gamma)\, x\right)$, which dominates the relaxation process.

In the presence of the nonlocal term, the solution of Eq. (1) is formally given by Eq. (5); however, the Green function for this case is given by

$$\mathcal{G}_{\text{nonlocal}}(\mathbf{r}, \mathbf{r}'; t) = \mathcal{G}(\mathbf{r}, \mathbf{r}'; t) + \sum_{n=1,l=0}^{\infty} \psi(\mathbf{r}', k_{n,l})\psi(\mathbf{r}, k_{n,l})\Phi\left(t, k_{n,l}\right), \tag{12}$$

with

$$\Phi\left(t, k_{n,l}\right) = \sum_{k=1}^{\infty} \frac{(-1)^n}{\Gamma(1+n)} \int_0^t dt_n \alpha(t-t_n) \int_0^{t_n} dt_{n-1}\alpha(t_n - t_{n-1}) \cdots \int_0^{t_2} dt_1 \alpha(t_2 - t_1) t_1^{\gamma n} \mathrm{E}_\gamma^{(n)}\left(-\mathcal{D}k_{n,l}^2 t^\gamma\right). \tag{13}$$

Let us extend the previous results by considering the unconfined region, i.e., a free diffusion, where the surface effects are not relevant. In order to investigate this scenario, we first consider, for simplicity, the situation characterized by radial symmetry and, then, the angular dependence is incorporated. In the first scenario, the system only depends on the variable r. One may use the eigenfunctions of the Sturm–Liouville problem related to the spatial operator as in the previous case, with the same boundary conditions for ρ. For this case, one finds

$$\mathcal{G}(r, r'; s) = \int_0^\infty dk\, k\, \varphi(r, k)\, \overline{\mathcal{G}}(k, r'; s), \tag{14}$$

whose inverse is given by

$$\overline{\mathcal{G}}(k, r'; s) = \int_0^\infty dr\, r^{\alpha-1}\, \varphi(r, k)\, \mathcal{G}(r, r'; s), \tag{15}$$

where $\varphi(r, k) = r^{1-\alpha/2}\mathrm{J}_\nu(kr)$, $\nu = \mathcal{N}_f/2 - 1$, and $\mathrm{J}_\nu(x)$ is the Bessel function. It is possible to show that

$$\overline{\mathcal{G}}(k, r'; s) = \frac{\varphi(r', k)}{s + \mathcal{D}s^{1-\gamma}k^2}. \tag{16}$$

The inverse Laplace transform of Eq. (16) yields

$$\overline{\mathcal{G}}(k, r'; t) = \varphi(r', k)\mathrm{E}_\gamma\left(-\mathcal{D}k^2 t^\gamma\right). \tag{17}$$

Note the presence of the Mittag-Leffler function [49] in the previous equation which implies an anomalous spreading of the initial condition as the case worked out before. By using the previous results, one finds that the propagator is given by

$$\mathcal{G}(r, r'; t) = \int_0^\infty dk k \varphi(r', k)\varphi(r, k)\mathrm{E}_\gamma\left(-\mathcal{D}k^2 t^\gamma\right). \tag{18}$$

In particular, it can be written as

$$\mathcal{G}(r, r'; t) = \frac{rr'}{(rr')^{\mathcal{N}_f/2}} \mathrm{H}^{1,0,1,1,1}_{2,[0:1],0,[0:2]}\left[\begin{matrix}(r'/r)^2 \\ \mathcal{D}t^\gamma/r^2\end{matrix}\left|\begin{matrix}\left(\frac{2-\nu}{2}, 1\right); \left(\frac{2+\nu}{2}, 1\right) \\ --; (0,1) \\ \left(-\frac{\nu}{2}, 1\right); \left(\frac{\nu}{2}, 1\right); (0,1), (0,\gamma)\end{matrix}\right.\right], \tag{19}$$

where

$$\mathrm{H}^{L,M,M_1,N,N_1}_{E,[A:C],F,[B,D]}\left[\begin{matrix}x \\ y\end{matrix}\left|\begin{matrix}(\varepsilon_1, \omega_1), \ldots, (\varepsilon_E, \omega_E) \\ (a_1, \alpha_1), \ldots, (a_A, \alpha_A); (c_1, \gamma_1), \ldots, (c_C, \gamma_C) \\ (\xi_1, \varpi_1), \ldots, (\xi_F, \varpi_F) \\ (b_1, \beta_1), \ldots, (b_B, \beta_B); (d_1, \delta_1), \ldots, (d_C, \delta_D)\end{matrix}\right.\right] \tag{20}$$

is the generalized Fox H function [56,57]. For the particular case $\gamma = 1$, Eq. (18) can be simplified to

$$\mathcal{G}(r, r'; t) = \frac{1}{2\mathcal{D}t}(rr')^{1-\mathcal{N}_f/2} e^{-\frac{1}{4\mathcal{D}t}(r^2+r'^2)} \mathrm{I}_\nu\left(\frac{rr'}{2\mathcal{D}t}\right). \tag{21}$$

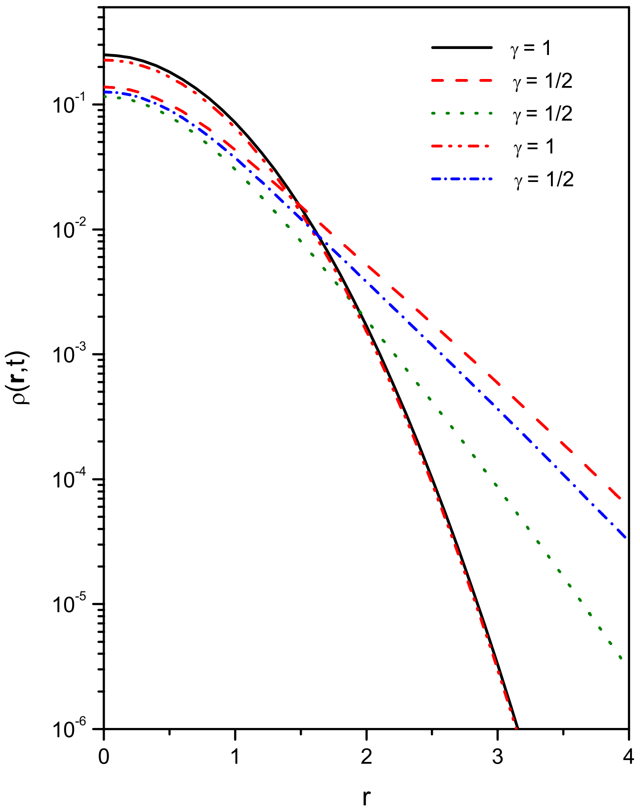

Fig. 1. The behavior of $\rho(\mathbf{r},t)$ versus r is illustrated by considering different time dependences for the nonlocal term. The red dashed–dotted line corresponds to the usual case, i.e., $\gamma = 1$ with $\alpha(t) = \alpha\delta(t)$. The black solid line is the usual diffusion equation in the presence of the nonlocal term with $\alpha(t) = \alpha e^{-\eta t}$. The blue short dashed–dotted line corresponds to the fractional case with $\alpha(t) = \alpha\delta(t)$. The red dashed line and the green dotted line are the fractional case with $\alpha(t) = \alpha t^{\mu-1}/\Gamma(\mu)$ with $\mu = 1/2$ and $\mu = -1/2$. For simplicity, $\mathcal{D} = 1, \xi = 0.1, \alpha = 1, t = 0.1$, and $\mathcal{N}_f = 3$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

By incorporating the angular variable in previous calculations, the propagator becomes

$$\mathcal{G}(\mathbf{r},\mathbf{r}';t) = \sum_{l=0}^{\infty}\int_0^{\infty} dpp\Theta_l(\theta)\Theta_l(\theta')\varphi_l(r',p)\varphi_l(r,p)\mathrm{E}_\gamma\left(-\mathcal{D}p^2t^\gamma\right), \tag{22}$$

with

$$\Theta_l(\theta) = \mathcal{N}_l C_l^{\mathcal{N}_f/2-1}(\cos\theta), \tag{23}$$

where $C_l^{\tilde{\alpha}}(\cos\theta)$ are the Gegenbauer polynomials [54], and

$$\mathcal{N}_l^2 = \frac{l!\left(l+\frac{\mathcal{N}_f}{2}-1\right)}{2^{3-\mathcal{N}_f}\pi\Gamma\left(l+\mathcal{N}_f-2\right)}\left[\Gamma\left(\frac{\mathcal{N}_f}{2}-1\right)\right]^2. \tag{24}$$

The functions $\varphi_l(r,p)$ are defined as before, with $\nu = \alpha/2+l-1$. The solution of Eq. (5), after inverting the Laplace transform, can be written in terms of the Green function as follows:

$$\rho(\mathbf{r}',t) = -\int_0^{\infty} drr^{\mathcal{N}_f-1}\int_0^{\pi} d\theta\,\sin^{\mathcal{N}_f-2}\theta\,\rho(\mathbf{r}',0)\mathcal{G}(\mathbf{r},\mathbf{r}';t). \tag{25}$$

Similar to the previous case, the presence of the nonlocal term changes only the time behavior of the Green function. In fact, after performing some calculations, it is possible to show that

$$\mathcal{G}_{\text{nonlocal}}(\mathbf{r},\mathbf{r}';t) = \mathcal{G}(\mathbf{r},\mathbf{r}';t) + \sum_{l=0}^{\infty}\int_0^{\infty} dpp\Theta_l(\theta)\Theta_l(\theta')\varphi_l(r',p)\varphi_l(r,p)\Phi(t,p), \tag{26}$$

with $\Phi(t,p)$ defined by Eq. (13), where $k_{n,l}$ is replaced by p. By using Eq. (25), we illustrate in Fig. 1 the influence of the fractional derivative and the nonlocal term on the spreading of the initial condition $\rho(\mathbf{r},0) \propto e^{-\xi\mathbf{r}^2}$. Note that, for the case $\gamma \neq 1$, the asymptotic behavior is different from the usual one. In Fig. 2, we show the behavior of the second moment for the initial condition $\rho(\mathbf{r},0) \propto \delta(\mathbf{r})$.

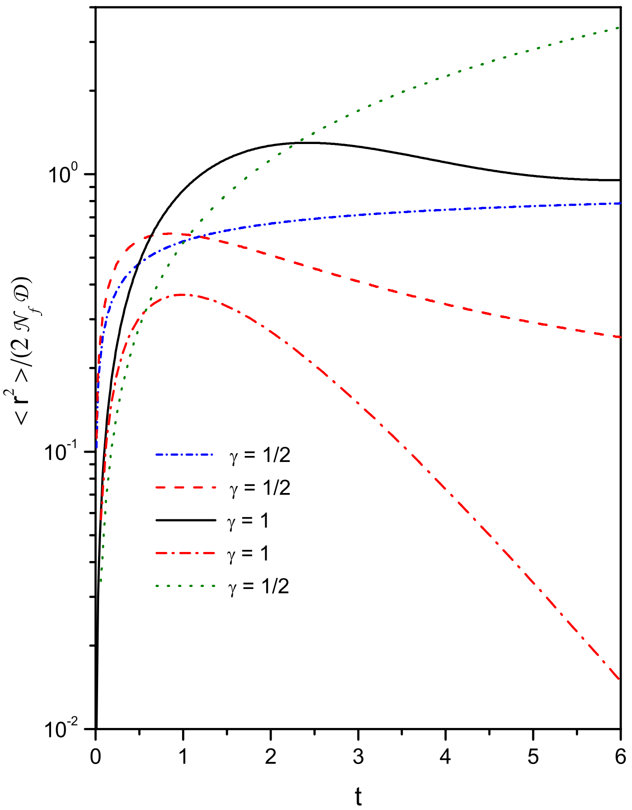

Fig. 2. The behavior of $\langle r^2\rangle/(2\mathcal{N}_f\mathcal{D})$ versus t is illustrated for different values of γ by considering the same time dependences for the nonlocal term presented in Fig. 1. For simplicity, $\mathcal{D} = 1$ and $\alpha = 1$.

3. Discussions and conclusions

We investigate the solutions of a fractional diffusion with noninteger dimensions in the presence of a nonlocal term, by first considering the system in a confined region with inhomogeneous time dependent boundary conditions. Next, we extend the previous solution for an unlimited region. In both cases, the solutions present an anomalous spreading of the initial condition which may be connected to anomalous diffusion. In Figs. 1 and 2, we have illustrated the spreading of a system initially characterized by a Gaussian distribution and the behavior of the second moment when the initial condition of the system is a delta function. These figures show that the system's behavior for both initial conditions is different from the usual. Finally, we hope that the results found here can be useful for discussing situations connected to anomalous diffusion.

Acknowledgments

We would like to thank the CNPq/INCT-SC and the Fundação Araucária (Brazilian agencies) for the financial support.

References

[1] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Physics Reports 339 (1) (2000) 1–77.
[2] W.T. Coffey, Yu.P. Kalmykov, J.T. Waldron, The Langevin Equation: With Applications to Stochastic Problems in Physics, Chemistry and Electrical Engineering, World Scientific Publishing Company, Singapore, 2004.
[3] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[4] B. O'Shaughnessy, I. Procaccia, Analytical solutions for diffusion on fractal objects, Physical Review Letters 54 (1985) 455–458.
[5] R. Metzler, W.G. Glöckle, T.F. Nonnenmacher, Fractional model equation for anomalous diffusion, Physica A 211 (1994) 13–24.
[6] D.V. Nicolau, J.F. Hancock, K. Burrage, Sources of anomalous diffusion on cell membranes: a Monte Carlo study, Biophysical Journal 92 (6) (2007) 1975–1987.
[7] G. Guigas, M. Weiss, Sampling the cell with anomalous diffusion—the discovery of slowness, Biophysical Journal 94 (2008) 90–94.
[8] A.M. Edwards, R.A. Phillips, N.W. Watkins, M. Freeman, E. Murphy, V. Afanasyev, S. Buldyrev, M. da Luz, E. Raposo, H. Stanley, G. Viswanathan, Revisiting Lévy flight search patterns of wandering albatrosses, bumblebees and deer, Nature 449 (2007) 1044–1048.
[9] D. Wanduku, G.S. Ladde, Global properties of a two-scale network stochastic delayed human epidemic dynamic model, Nonlinear Analysis. Real World Applications 13 (2012) 794–816.
[10] N. Yang, G. Zhang, B.W. Li, Violation of Fourier's law and anomalous heat diffusion in silicon nanowires, Nano Today 5 (2010) 85–90.
[11] S. Ozturk, Y.A. Hassan, V.M. Ugaz, Interfacial complexation explains anomalous diffusion in nanofluids, Nano Letters 10 (2010) 665–671.
[12] V.R. Voller, C. Paola, Can anomalous diffusion describe depositional fluvial profiles? Journal Geophysical Reserach-Earth Surfurface 115 (2010) F00A13.

[13] V. Ganti, M.M. Meerschaert, E. Foufoula-Georgiou, E. Viparelli, G. Parker, Normal and anomalous diffusion of gravel tracer particles in rivers, Journal of Geophysical Research-Earth Surface 115 (2010) F00A12.
[14] P. Tierno, T.H. Johansen, T.M. Fischer, Localized and delocalized motion of colloidal particles on a magnetic bubble lattice, Physical Review Letters 99 (2007) 038303.
[15] V. Latora, A. Rapisarda, S. Ruffo, Superdiffusion and out-of-equilibrium chaotic dynamics with many degrees of freedoms, Physical Review Letters 83 (11) (1999) 2104.
[16] V. Latora, A. Rapisarda, C. Tsallis, Non-Gaussian equilibrium in a long-range Hamiltonian system, Physical Review E 64 (5) (2001) 056134.
[17] M. Giona, M. Giustiniani, Adsorption kinetics on fractal surfaces, The Journal of Physical Chemistry 100 (41) (1996) 16690–16699.
[18] E.K. Lenzi, L.R. Evangelista, G. Barbero, Anomalous diffusion and the adsorption–desorption process in anisotropic media, Europhysics Letters 85 (2) (2009) 28004.
[19] E.K. Lenzi, C.A.R. Yednak, L.R. Evangelista, Non-Markovian diffusion and the adsorption–desorption process, Physical Review E 81 (1) (2010) 011116.
[20] G.H. Weiss, Aspects and Applications of the Random Walk, North-Holland, Amsterdam, 1994.
[21] I. Oppenheim, K.E. Shuler, G.H. Weiss (Eds.), Stochastic Processes in Chemical Physics: The Master Equation, MIT Press, Cambridge, Massachusetts, 1977.
[22] B. Ahmad, J.J. Nieto, A. Alsaedi, M. El-Shahed, A study of nonlinear Langevin equation involving two fractional orders in different intervals, Nonlinear Analysis. Real World Applications 13 (2012) 599–606.
[23] Y. Luchko, Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation, Journal of Mathematical Analysis and Applications 374 (2011) 538–548.
[24] M. Yang, P.E. Kloeden, Random attractors for stochastic semi-linear degenerate parabolic equations, Nonlinear Analysis. Real World Applications 12 (2011) 2811–2821.
[25] R. Metzler, J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, Journal of Physics A 37 (31) (2004) R161–R208.
[26] Y.Z. Povstenko, Fractional heat conduction equation and associated thermal stress, Journal of Thermal Stresses 28 (1) (2005) 83–102.
[27] F. Mainardi, The fundamental solutions for the fractional diffusion-wave equation, Applied Mathematics Letters 9 (6) (1996) 23–28.
[28] W. Wyss, The fractional diffusion equation, Journal of Mathematical Physics 27 (11) (1986) 2782–2785.
[29] W.R. Schneider, W. Wyss, Fractional diffusion and wave equations, Journal of Mathematical Physics 30 (1) (1989) 134–144.
[30] Y. Fujita, Integrodifferential equation which interpolates the heat equation and the wave equation, Osaka Journal of Mathematics 27 (2) (1990) 309–321.
[31] A. Hanyga, Multidimensional solutions of time-fractional diffusion-wave equations, The Proceedings of the Royal Society of London. Series A 458 (2020) (2002) 933–957.
[32] Y.Z. Povstenko, Fractional radial diffusion in a cylinder, Journal of Molecular Liquids 137 (1–3) (2008) 46–50.
[33] R. Rossato, M.K. Lenzi, L.R. Evangelista, E.K. Lenzi, Fractional diffusion equation in a confined region: surface effects and exact solutions, Physical Review E 76 (3) (2007) 032102.
[34] E.K. Lenzi, R. Rossato, M.K. Lenzi, L.R. da Silva, G. Goncalves, Fractional diffusion equation and external forces: solutions in a confined region, Zeitschrift für Naturforschung—Section A 65 (5) (2010) 423–430.
[35] X. Jiang, M. Xu, The time fractional heat conduction equation in the general orthogonal curvilinear coordinate and the cylindrical coordinate systems, Physica A 389 (17) (2010) 3368–3374.
[36] Y.Z. Povstenko, Central symmetric solution to the Neumann problem for a time-fractional diffusion-wave equation in a sphere, Nonlinear Analysis: Real World Applications 13 (3) (2012) 1229–1238.
[37] N. Laskin, Fractional Schrödinger equation, Physical Review E 66 (2002) 056108.
[38] M. Naber, Exact solutions of fractional Schödinger-like equation with a nonlocal term, Journal of Mathematical Physics 45 (2004) 3339.
[39] M. Jeng, S.-L.-Y. Xu, E. Hawkins, J.M. Schwarz, On the nonlocality of the fractional Schrödinger equation, Journal of Mathematical Physics 51 (2010) 062102.
[40] E.C. de Oliveira, F.S. Costa, J. Vaz Jr., The fractional Schrödinger equation for delta potentials, Journal of Mathematical Physics 51 (2010) 123517.
[41] H. Ertik, D. Demirhan, H. Sirin, F. Büyükkiliç, Time fractional development of quantum systems, Journal of Mathematical Physics 51 (2010) 082102.
[42] S. Wang, M. Xu, X. Li, Green's function of time fractional diffusion equation and its applications in fractional quantum mechanics, Nonlinear Analysis. Real World Applications 10 (2009) 1081–1086.
[43] E.K. Lenzi, H.V. Ribeiro, H. Mukai, R.S. Mendes, Continuous-time random walk as a guide to fractional Schrödinger equation, Journal of Mathematical Physics 51 (2010) 092102.
[44] F.H. Stillinger, Axiomatic basis for spaces with noninteger dimension, Journal of Mathematical Physics 18 (1977) 1224.
[45] R. Eid, S.I. Muslih, D. Baleanu, E. Rabei, On fractional Schrödinger equation in α—dimensional fractional space, Nonlinear Analysis. Real World Applications 10 (2009) 1299–1304.
[46] E.K. Lenzi, B.F. de Oliveira, L.R. da Silva, L.R. Evangelista, Solutions for a Schrödinger equation with a nonlocal term, Journal of Mathematical Physics 49 (2008) 032108.
[47] X. Jiang, H. Qi, M. Xu, Exact solutions of fractional Schrödinger-like equation with a nonlocal term, Journal of Mathematical Physics 52 (2011) 042105.
[48] J. Dong, Generalized Lippmann–Schwinger equation in the fractional quantum mechanics, Journal of Physics A: Mathematical and Theoretical 44 (2011) 215204.
[49] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[50] M. Zubair, M.J. Mughal, Q.A. Naqvi, On electromagnetic wave propagation in fractional space, Nonlinear Analysis. Real World Applications 12 (2011) 2844–2850.
[51] A. Schot, M.K. Lenzi, L.R. Evangelista, L.C. Malacarne, R.S. Mendes, E.K. Lenzi, Fractional diffusion equation with an absorbent term and a linear external force: Exact solution, Physics Letters A 366 (2007) 346–350.
[52] E.K. Lenzi, M.K. Lenzi, R. Rossato, L.R. Evangelista, L.R. da Silva, Results for a fractional diffusion equation with a nonlocal term in spherical symmetry, Physics Letters A 372 (40) (2008) 6121–6124.
[53] E.K. Lenzi, H.V. Ribeiro, J. Martins, M.K. Lenzi, G.G. Lenzi, S. Specchia, Non-Markovian diffusion equation and diffusion in a porous catalyst, Chemical Engineering Journal 172 (2011) 1083–1087.
[54] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, San Diego, 2007.
[55] X. Jiang, M. Xu, The fractional finite Hankel transform and its applications in fractal space, Journal of Physics A: Mathematical and Theoretical 42 (2009) 385201.
[56] X. Jiang, M. Xu, The time fractional heat conduction equation in the general orthogonal curvilinear coordinate and the cylindrical coordinate systems, Physica A 389 (17) (2010) 3368–3374.
[57] A.M. Mathai, R.K. Saxena, The H-Function with Application in Statistics and Other Disciplines, Wiley Eastern, New Delhi, 1978.