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Exact propagator for a Fokker-Planck equation, first passage time distribution, and anomalous diffusion

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We obtain an exact form for the propagator of the Fokker-Planck equation $\partial_t \rho = \partial_x (\mathcal{D}(x)\partial_x \rho) - \partial_x (F(x,t)\rho)$, with $\mathcal{D}(x) = \widetilde{\mathcal{D}}|x|^{-\eta}$ in presence of the external force $F(x,t) = -k(t)x + (\mathcal{K}/x)\,|x|^{-\eta}$. Using the results found here, we also investigate the mean square displacement, survival probability, and first passage time distribution. In addition, we discuss the connection of these results with anomalous diffusion phenomena. © *2011 American Institute of Physics*. [doi:10.1063/1.3621823]

I. INTRODUCTION

A typical diffusion process has, as characteristic, a linear time dependence on the mean square displacement,[1,2] i.e., $\langle (x - \langle x \rangle)^2 \rangle \sim t$, which is related to the Markovian nature of this stochastic process. However, there are several physical situations, such as diffusion of high molecular weight polyisopropylacrylamide in nanopores,[3] highly confined hard disk fluid mixture,[4] fluctuating particle fluxes,[5] diffusion on fractals,[6,7] ferrofluid,[8] nanoporous material,[9] diffusion on disordered media,[10] diffusion of grains,[11] and colloids,[12] present a different behavior for the mean square displacement and in some cases it is given by $\langle (x - \langle x \rangle)^2 \rangle \sim t^{\alpha}$. To face these phenomena, which exhibit an anomalous diffusion, several approaches have been used. For instance, Langevin equations,[13–16] random walks,[17] Fokker-Planck equations,[18] and extensions by considering nonlinear terms[19] or fractional derivatives.[20–23] The formal aspects of these formalism has also been analyzed by considering several situations, such as presence of external forces,[24,25] reaction terms,[26–28] and variable boundary conditions,[29] in order to comprehend the formalisms and their potential applications. In this direction, we investigate the solutions of the Fokker-Planck equation

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{\partial}{\partial x}\left[\mathcal{D}(x)\frac{\partial}{\partial x}\rho(x,t) - F(x,t)\rho(x,t)\right], \tag{1}$$

where the diffusion coefficient is given by $\mathcal{D}(x) = \widetilde{\mathcal{D}}|x|^{-\eta}$ ($\widetilde{\mathcal{D}} = \text{const.}$) and $F(x,t)$ is the external force (*drift*) associated with the potential $V(x,t) = k(t)x^2/2 + (\mathcal{K}/\eta)(1/|x|^{\eta} - 1)$. The potential employed here can be considered as an extension of the logarithmic potential used, for instance, to establish the connection between the fractional diffusion coefficient and the generalized mobility.[30] Note that the range of η values is defined according to the situation analyzed and the procedure of calculation used to obtain the solution. This analysis is performed in Sec. II for each situation investigated. The external force obtained from it has, as particular cases, the Ornstein-Uhlenbeck[1,2] and the Rayleigh[1] processes. It also recovers the external force considered in Refs. 31–34, when $k(t) = \overline{k} = \text{constant}$, with $\mathcal{K} = 0$, and the situations worked out in Ref. 35 for $\eta = 0$ with $\mathcal{K} = 0$. Furthermore, Eq. (1) may be used to investigate several physical situations such as diffusion on

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fractals,[7,36] turbulence,[37] fast electrons in a hot plasma in the presence of a electric field,[38] financial models,[39,40] and non-Markovian random walk.[41] In addition, Ref. 42 has investigated a similar equation to Eq. (1) with $\mathcal{D}(x)$ algebraic in connection with a Markovian and coupled version of the continuous time random walk theory.[43]

Our goal is to investigate solutions, mean square displacement, and the first passage time distribution for Eq. (1). We initially consider the case characterized by the harmonic potential and, after that, incorporate the additional power law term in the potential. The solutions are obtained by considering the boundary condition $\rho(\pm\infty, t) = 0$ and the arbitrary initial condition $\rho(x, 0) = \tilde{\rho}(x)$. These solutions are used to investigate the mean square displacement. In particular, we show that for suitable choice of $k(t)$ it is possible to get a power law behavior for it and to extend results found in Refs. 32 and 33 for a time-independent external force. A situation with two different diffusive regimes is also considered. It is worth noting that the solution of Eq. (1), for the previous boundary condition, in absence of external force, is essentially a stretched exponential, i.e., $\rho(x, t) \propto e^{-|x|^{2+\eta}/((2+\eta)^2\widetilde{\mathcal{D}}t)}/t^{1/(2+\eta)}$, and consequently $\langle (x - \langle x\rangle)^2\rangle \propto t^{2/(2+\eta)}$ which can be related to a sub- or superdiffusive process depending on the choice of η. The first passage time distribution is analyzed by considering the system subjected to the boundary conditions $\rho(0, t) = 0$ and $\rho(\infty, t) = 0$ and accomplishing the condition $\lim_{t\to\infty} \rho(x, t) = 0$. These developments are performed in Sec. II, and in Sec. III the discussions and conclusions are presented.

II. FOKKER-PLANCK EQUATION AND PROPAGATOR

Let us investigate the solutions of Eq. (1) by taking the external force $F(x, t) = -k(t)x + (\mathcal{K}/x)|x|^{-\eta}$ into account. We first analyze the case characterized by a linear external force, i.e., $F(x, t) = -k(t)x$ (where $k(t)$ is a non-negative function); subsequently, the case $F(x, t) = -k(t)x + (\mathcal{K}/x)|x|^{-\eta}$ is considered.

A. External force: $F(x, t) = -k(t)x$

In order to find the propagator (or Green function) for Eq. (1), subjected to the external force $F(x, t) = -k(t)x$, we consider the boundary condition $\rho(\pm\infty, t) = 0$ and an arbitrary initial condition as previously discussed in Sec. I. For this case, Eq. (1) can be written as

$$\frac{\partial}{\partial t}\rho(x, t) = \frac{\partial}{\partial x}\left[\widetilde{\mathcal{D}}|x|^{-\eta}\frac{\partial}{\partial x}\rho(x, t) + k(t)\,x\,\rho(x, t)\right]. \tag{2}$$

Note that the solutions of this equation have, as important limiting cases, the situations worked out in Refs. 33 and 35. To avoid cumbersome calculations, following the scenarios worked out in Ref. 44, we perform the change of variable $x \to z = \xi(t)x$ and $t \to \beta(t)$, with $\rho(x, t) \equiv \rho(z, \beta)$. Before implementing this change of variables, we expand the last term of Eq. (2) and consider the transformation $\rho(x, t) = \exp\left(\int_0^t d\bar{t}k(\bar{t})\right)\overline{\rho}(x, t)$. This procedure leads us to

$$\frac{\partial}{\partial t}\overline{\rho}(x, t) = \widetilde{\mathcal{D}}\frac{\partial}{\partial x}\left[|x|^{-\eta}\frac{\partial}{\partial x}\overline{\rho}(x, t)\right] + k(t)x\frac{\partial}{\partial x}\overline{\rho}(x, t)\,. \tag{3}$$

Now, after performing in Eq. (3) the change of variables just proposed, we obtain

$$\frac{\partial}{\partial \beta}\overline{\rho}(z, \beta) = \widetilde{\mathcal{D}}\frac{\partial}{\partial z}\left[|z|^{-\eta}\frac{\partial}{\partial z}\overline{\rho}(z, \beta)\right], \tag{4}$$

where $z = \xi(t)x$, $\beta(t) = \int_0^t dv \exp[(2+\eta)\int_0^v du k(u)]$, and $\xi(t) = \exp\left[\int_0^t du k(u)\right]$. The propagator of Eq. (4) (and consequently of Eq. (2)) may be found by using the functions

$$\Psi_+(z, \lambda) = |z|^{\frac{1}{2}(1+\eta)}\mathrm{J}_{-\nu}\left(\frac{2\lambda}{2+\eta}|z|^{\frac{1}{2}(2+\eta)}\right) \quad \text{and} \tag{5}$$

$$\Psi_-(z, \lambda) = z|z|^{\frac{1}{2}(\eta-1)}\mathrm{J}_{\nu}\left(\frac{2\lambda}{2+\eta}|z|^{\frac{1}{2}(2+\eta)}\right), \tag{6}$$

which are solutions of the spatial operator present in Eq. (4). In Eqs. (5) and (6), the sign + and − refer, respectively, to the even and odd functions, $\nu = (1+\eta)/(2+\eta)$ and λ is a constant (eigenvalue) related to the Sturm-Liouville problem of the spatial operator,[45] i.e., $\partial_z(|z|^{-\eta}\partial_z\Psi_\pm(z)) = -\lambda^2\Psi_\pm(z)$, present in Eq. (4). By using Eqs. (4) and (6), the solution for Eq. (4) may be written as

$$\overline{\rho}(z,\beta) = \int_0^\infty d\lambda\,[\mathcal{C}_+(\lambda,\beta)\Psi_+(z,\lambda) + \mathcal{C}_-(\lambda,\beta)\Psi_-(z,\lambda)]\,, \tag{7}$$

where $\mathcal{C}_+(\lambda,\beta)$ and $\mathcal{C}_-(\lambda,\beta)$ are time-dependent functions, which can be found by substituting Eq. (7) into Eq. (2) and using the orthogonality properties of the function $\Psi_+(z)$ and $\Psi_-(z)$. After some calculations, it is possible to show that

$$\mathcal{C}_\pm(\lambda,\beta) = \mathcal{C}_\pm(\lambda,0)e^{-\widetilde{\mathcal{D}}\beta\lambda^2}, \tag{8}$$

with

$$\mathcal{C}_\pm(\lambda,0) = \frac{\lambda}{2+\eta}\int_{-\infty}^{\infty} dx\rho(x,0)\Psi_\pm(x,\lambda), \tag{9}$$

since $\xi(0)=1$ and, consequently, $z=x$ for $t=0$. The range of values of η is defined, in the approach used here, by the integrals involving the Bessel functions which emerge from orthogonality condition and, consequently, by the initial condition, which implies in the eigenfunction used to get the solution. Possible situations are: (i) the initial condition given by an odd function which leads us to the range $-4/3 < \eta$; (ii) the initial condition given by an even function or a mixing between an even and odd functions; for this case, the range of values of η is $-4/3 < \eta \leq 0$. Substitution of Eqs. (8) and (9) into Eq. (7), yields $\rho(z,\beta) = \int_{-\infty}^{\infty} d\overline{x}\rho(\overline{x},0)\mathcal{G}(z,\overline{x},\beta)$, with

$$\overline{\mathcal{G}}(z,\overline{x},\beta) = \frac{\xi}{2(2+\eta)\widetilde{\mathcal{D}}\beta}|z\overline{x}|^{\frac{1}{2}(1+\eta)}e^{-|z|^{2+\eta}/[(2+\eta)^2\widetilde{\mathcal{D}}\beta]}e^{-|\overline{x}|^{2+\eta}/[(2+\eta)^2\widetilde{\mathcal{D}}\beta]}$$
$$\times\left\{\mathrm{I}_{-\nu}\left[\frac{2}{(2+\eta)^2\widetilde{\mathcal{D}}\beta}|z\overline{x}|^{\frac{1}{2}(2+\eta)}\right] + \frac{z\overline{x}}{|z\overline{x}|}\mathrm{I}_\nu\left[\frac{2}{(2+\eta)^2\widetilde{\mathcal{D}}\beta}|z\overline{x}|^{\frac{1}{2}(2+\eta)}\right]\right\}, \tag{10}$$

where $z=\xi(t)x$, and the functions $\xi(t)$ and $\beta(t)$ defined after Eq. (4). It is important to emphasize that $\overline{\mathcal{G}}(x,\overline{x},t) \equiv \overline{\mathcal{G}}(z,\overline{x},\beta)$ after substituting $z=\xi(t)x$ and the functions $\xi(t)$ and $\beta(t)$ into Eq. (10). Note that Eq. (10) recovers the result presented in Refs. 33 and 34 for a time-independent external force when $k(t)=\overline{k}=$ constant. By using the previous results, it is possible to calculate the mean square displacement to this process, which is governed by Eq. (10). To perform this calculation, we consider, for simplicity, the initial condition given by $\rho(x,0)=\delta(x)$. For this initial condition, the solution is

$$\rho(x,t) = \frac{\xi(t)e^{-(\xi(t)|x|)^{2+\eta}/[(2+\eta)^2\widetilde{\mathcal{D}}\beta(t)]}}{2\Gamma\left(\frac{3+\eta}{2+\eta}\right)\left[(2+\eta)^2\widetilde{\mathcal{D}}\beta(t)\right]^{\frac{1}{2+\eta}}}, \tag{11}$$

and, for the mean square displacement, i.e., $(\Delta x)^2 = \langle(x-\langle x\rangle)^2\rangle$, we obtain

$$(\Delta x)^2 = \frac{\Gamma\left(\frac{3}{2+\eta}\right)}{\Gamma\left(\frac{1}{2+\eta}\right)\xi^2(t)}\left((2+\eta)^2\widetilde{\mathcal{D}}\beta(t)\right)^{\frac{2}{2+\eta}}\,. \tag{12}$$

In this manner, for the initial condition considered above, the mean square displacement is equal to the second moment. Figure 1 illustrates the behavior of Eq. (12) for different choices of the time-dependent functions $k(t)$. In particular, for $k(t)=\overline{k}/(1+t)$ the behavior of Eq. (12) is a power law similar to the ones found in anomalous diffusion. To check this feature, after some calculations we obtain

$$(\Delta x)^2 = \frac{\Gamma\left(\frac{3}{2+\eta}\right)}{\Gamma\left(\frac{1}{2+\eta}\right)(1+t)^{2\overline{k}}}\left[(2+\eta)^2\widetilde{\mathcal{D}}\frac{(1+t)^{(2+\eta)\overline{k}+1}-1}{(2+\eta)\overline{k}+1}\right]^{\frac{2}{2+\eta}}, \tag{13}$$

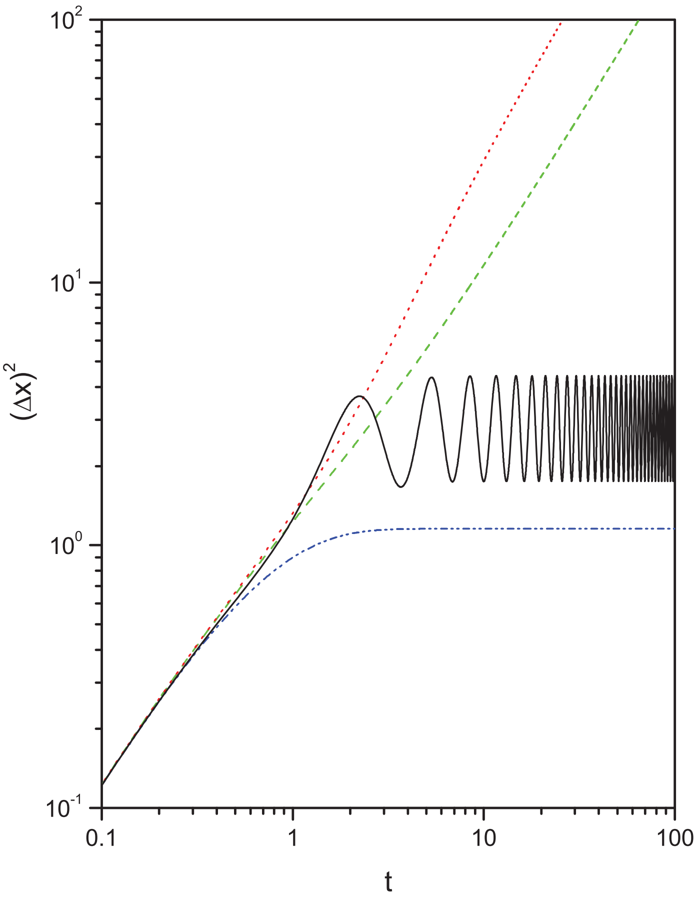

FIG. 1. (Color online) This figure illustrates Eq. (12) versus t for different time-dependent functions $k(t)$. The blue dashed-dotted line, which corresponds to the case $k(t) = \overline{k} = \text{const.}$, has a stationary value indicating the presence of a stationary solution for the system. The cases characterized by $k(t) = k/(1+t)$ and $k(t) = ke^{(-t)}$ are illustrated by the green dashed line and the red dotted line, respectively. These two cases for long time have as asymptotic behavior for the second moment a power law similar to the ones found in anomalous diffusion processes. The black solid line is the case $k(t) = \overline{k}\cos^2(t)$. For simplicity, in all cases we consider $\eta = -1/3$, $\overline{k} = 1$, and $\widetilde{\mathcal{D}} = 1$.

whose short or long time behavior is given by $(\Delta x)^2 \sim t^{\frac{2}{2+\eta}}$, i.e., it is a power law similar to the ones found in anomalous diffusion. The results found in Refs. 32 and 33 for time-independent external force verify a power law dependence to $(\Delta x)^2$ for short times but not for long times. Thus, the behavior exhibited by Eq. (13) indicates that the solution found above, i.e., Eq. (10), may be used to investigate situations related to anomalous diffusion depending on the choice of η and the time-dependent function $k(t)$ present in the external force.

B. External force: $F(x, t) = -k(t)x + (\mathcal{K}/x)|x|^{-\eta}$

It is possible to extend the results found above by incorporating a power law term to the potential. For this case, Eq. (1) can be written as

$$\frac{\partial}{\partial t}\rho(x,t) = \widetilde{\mathcal{D}}\frac{\partial}{\partial x}\left[|x|^{-\eta}\frac{\partial}{\partial x}\rho(x,t)\right] + \frac{\partial}{\partial x}\left[\left(k(t)x - \frac{\mathcal{K}}{x}|x|^{-\eta}\right)\rho(x,t)\right]. \tag{14}$$

Applying the procedure used to find the propagator for the previous case, Eq. (14) can be simplified to

$$\frac{\partial}{\partial \beta}\overline{\rho}(z,\beta) = \widetilde{\mathcal{D}}\frac{\partial}{\partial z}\left[|z|^{-\eta}\left(\frac{\partial}{\partial z}\overline{\rho}(z,\beta) - \frac{\mathcal{K}}{z}\overline{\rho}(z,t)\right)\right], \tag{15}$$

which has solution as

$$\overline{\rho}(z,\beta) = \int_0^\infty d\lambda \left[\mathcal{A}_+(\lambda,\beta)\overline{\Psi}_+(z,\lambda) + \mathcal{A}_-(\lambda,\beta)\overline{\Psi}_-(z,\lambda)\right], \tag{16}$$

with

$$\overline{\Psi}_+(z,\lambda) = |z|^{\frac{1}{2}(1+\eta+\mathcal{K}/\widetilde{\mathcal{D}})}\mathrm{J}_{-\overline{\nu}}\left[\frac{2\lambda}{2+\eta}|z|^{\frac{1}{2}(2+\eta)}\right] \quad \text{and} \tag{17}$$

$$\overline{\Psi}_-(z,\lambda) = z|z|^{\frac{1}{2}(\eta-1+\mathcal{K}/\widetilde{\mathcal{D}})}\mathrm{J}_{\overline{\nu}}\left[\frac{2\lambda}{2+\eta}|z|^{\frac{1}{2}(2+\eta)}\right], \tag{18}$$

where $\overline{\nu} = (1+\eta)/(2+\eta) - \mathcal{K}/[(2+\eta)\widetilde{\mathcal{D}}]$. The functions $\mathcal{A}_+(\lambda,\beta)$ and $\mathcal{A}_-(\lambda,\beta)$ can be found by substituting Eqs. (17) and (18) in Eq. (15) and using the orthogonality properties of these functions. In particular, they are given by

$$\mathcal{A}_\pm(\lambda,\beta) = \mathcal{A}_\pm(\lambda,0)e^{-\widetilde{\mathcal{D}}\beta\lambda^2}, \tag{19}$$

with

$$\mathcal{A}_\pm(\lambda,0) = \frac{\lambda}{2+\eta}\int_{-\infty}^{\infty} dx|x|^{-\mathcal{K}/\widetilde{\mathcal{D}}}\rho(x,0)\overline{\Psi}_\pm(x,\lambda)\,. \tag{20}$$

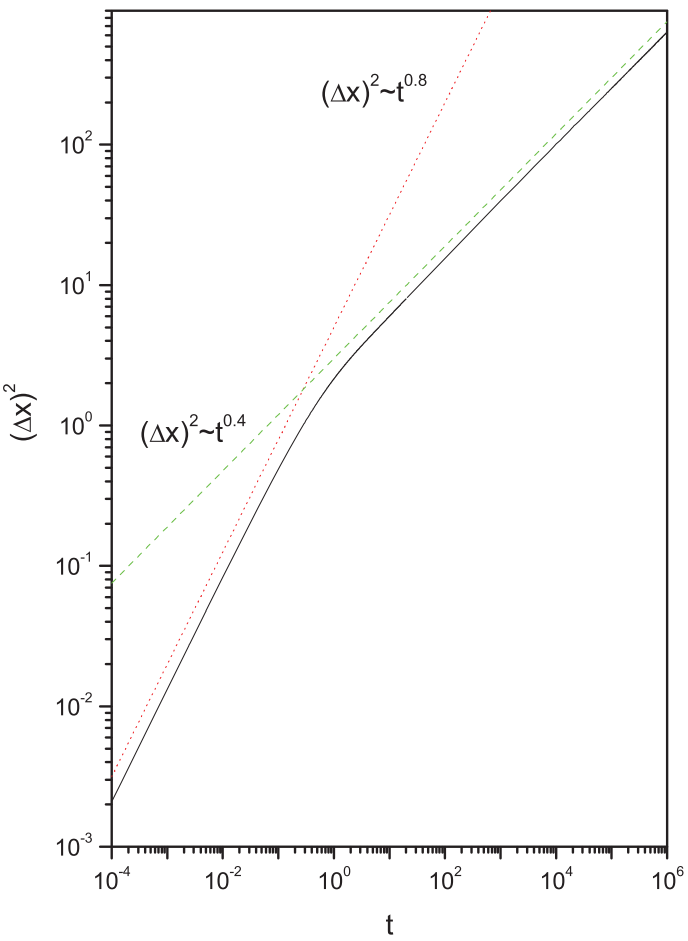

FIG. 2. (Color online) The behavior of $(\Delta x)^2$ versus t is illustrated, for simplicity, by considering $\eta = 1/2$, $\overline{k} = 1$, $\mathcal{K} = 1$, $\mathcal{D} = 1$, and the initial condition $\rho(x,0) = \delta(x)$. Note that the red dotted line and the dashed green line were incorporated in this figure to evidence the different diffusive regime obtained by considering $k(t) = \overline{k}\mathrm{E}_{1/2}\left(-t^{1/2}\right)$.

The range of values of η for this case also depends on the eigenfunctions used to get the solution. When the initial condition is given by an odd function, the range of values of η is given by $2\mathcal{K}/(3\widetilde{\mathcal{D}}) - 4/3 < \eta$. On the other hand, if the even eigenfunctions are present in the solution, the range of values of η is $2\mathcal{K}/(3\widetilde{\mathcal{D}}) - 4/3 < \eta \leq 2\mathcal{K}/(3\widetilde{\mathcal{D}})$. By substituting Eqs. (19) and (20) into Eq. (16), after some calculations we obtain the propagator in the form

$$\mathcal{G}(z, \overline{x}, \beta) = \frac{\xi\,|z\overline{x}|^{\frac{1}{2}(1+\eta)}}{2(2+\eta)\widetilde{\mathcal{D}}\beta|\overline{x}|^{\mathcal{K}/\widetilde{\mathcal{D}}}}|z\overline{x}|^{\mathcal{K}/(2\widetilde{\mathcal{D}})}e^{-|z|^{2+\eta}/[(2+\eta)^2\widetilde{\mathcal{D}}\beta]}e^{-|\overline{x}|^{2+\eta}/[(2+\eta)^2\widetilde{\mathcal{D}}\beta]}$$
$$\times\left\{\mathrm{I}_{-\overline{\nu}}\left[\frac{2}{(2+\eta)^2\widetilde{\mathcal{D}}\beta}|z\overline{x}|^{\frac{1}{2}(2+\eta)}\right] + \frac{z\overline{x}}{|z\overline{x}|}\mathrm{I}_{\overline{\nu}}\left[\frac{2}{(2+\eta)^2\widetilde{\mathcal{D}}\beta}|z\overline{x}|^{\frac{1}{2}(2+\eta)}\right]\right\}. \qquad (21)$$

Similar to the previous case, i.e., $\mathcal{K} = 0$, the solution here presents a mean square displacement which, depending on the choice of the time-dependent function $k(t)$, may present different trends and, in particular, the ones found in anomalous diffusion processes. In this direction, Fig. 2 shows that it is possible to get different diffusive regimes for a suitable choice of the term $k(t)$ in the external force. The time dependence chosen was $k(t) = \overline{k}\mathrm{E}_{1/2}(-t^{1/2})$, where $\mathrm{E}_\gamma(x)$ is the Mittag-Leffler function[46] which present two different regimes, one for short times and other one for long times.

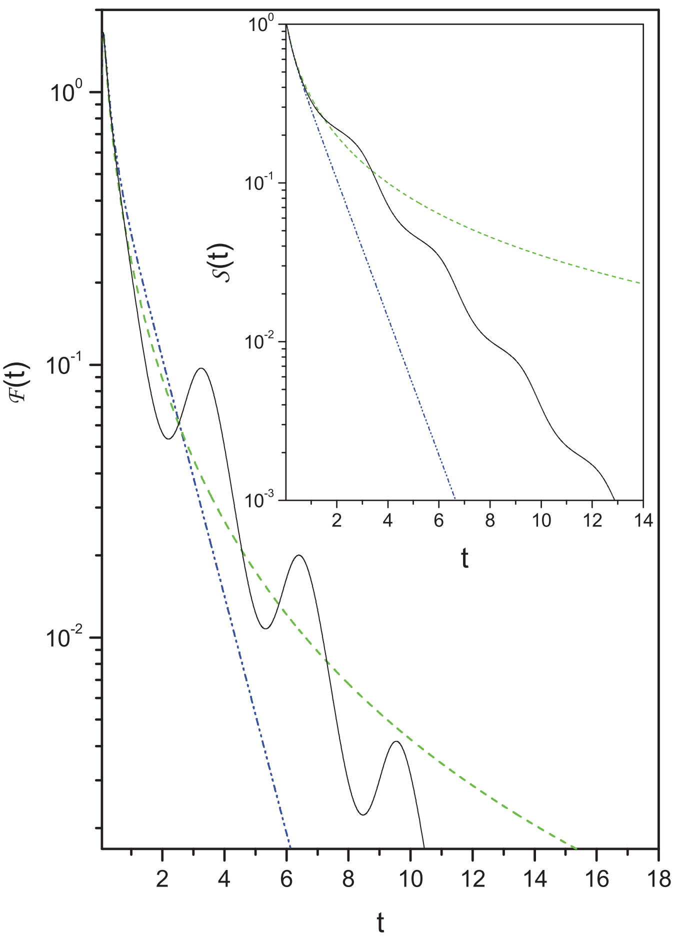

FIG. 3. (Color online) The behavior of Eq. (24) versus t for different time-dependent functions $k(t)$ is illustrated. The blue dashed-dotted line, green dashed line, and black line correspond, respectively, to the cases $k(t) = \overline{k}$, $k(t) = \overline{k}/(1+t)$, and $k(t) = \overline{k}\cos^2(t)$. The inset illustrates the behavior of Eq. (23) versus t. For simplicity, in all cases we consider $\eta = 1$, $\overline{x} = 1$, $\overline{k} = 1$, $\mathcal{K} = 1$, and $\mathcal{D} = 1$.

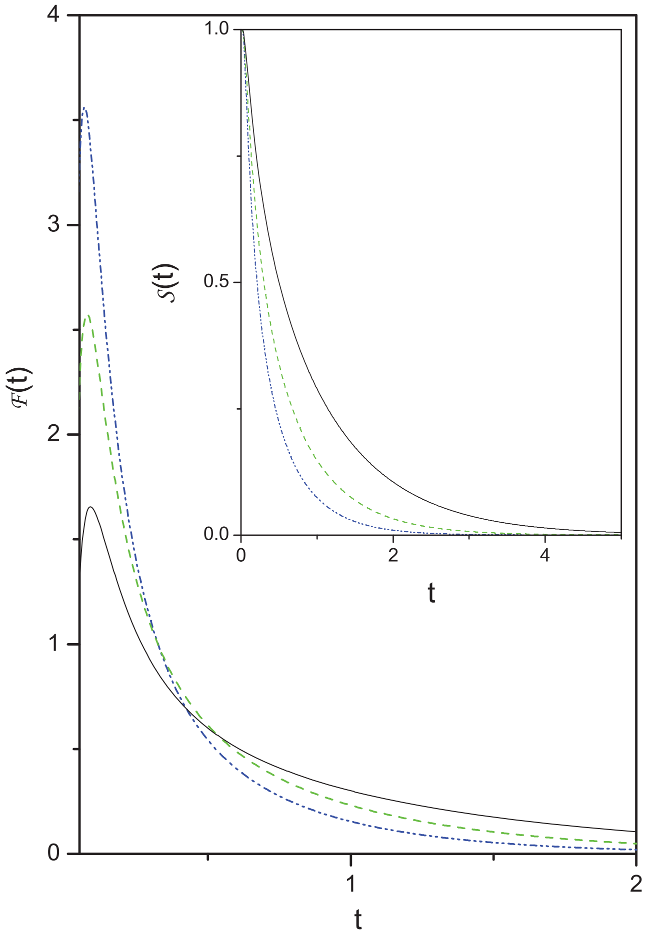

FIG. 4. (Color online) The behavior of Eq. (24) versus t for different time-dependent functions $\mathcal{K}$ is illustrated. The blue dashed-dotted line, green dashed line, and black line correspond, respectively, to the cases $\mathcal{K} = 0$, $\mathcal{K} = 0.5$, and $\mathcal{K} = 1.0$. The inset illustrates the behavior of Eq. (23) versus t. For simplicity, in all cases we consider $\eta = 1$, $\overline{x} = 1$, $\overline{k} = 1$, and $\mathcal{D} = 1$.

Let us analyze now the first passage time distribution for the system governed by Eq. (1) with the external force $F(x,t) = -k(t)x + (\mathcal{K}/x)|x|^{-\eta}$, which corresponds to the last case analyzed before. To avoid trivial solutions and obtain results with physical meaning, we consider the system subjected to boundary conditions $\rho(0,t) = 0$ and $\rho(\infty,t) = 0$ and the condition $\lim_{t\to\infty}\rho(x,t) = 0$. In this manner, the system is restricted to the non-negative half space with an absorbent surface removing the particles of the system at $x = 0$. The first passage time distribution relevant to this conditions has been investigated by considering different approaches, in particular, the fractional diffusion equations[47–49] and the Fokker-Planck equations with spatial and time dependence on the diffusion coefficient in absence of external forces.[50] In this direction, we use the solutions for the last case, i.e., Eq. (21), to obtain the first passage time distribution in presence of time-dependent external forces and, consequently, extend results found in Ref. 50. By using the previous results, it is possible to find the solution for the system subjected to the previous boundary conditions which characterizes a semi-infinite region. In particular, a solution which satisfies the conditions required, for simplicity, for the initial condition $\rho(x,0) = \delta(x - \overline{x})$, is given by

$$\rho(x,t) = \frac{\xi(t)\,(\xi(t)x\overline{x})^{\alpha}}{(2+\eta)\widetilde{\mathcal{D}}\beta(t)\overline{x}^{\mathcal{K}/\widetilde{\mathcal{D}}}} e^{-(\xi(t)x)^{2+\eta}/[(2+\eta)^2\widetilde{\mathcal{D}}\beta(t)]} \times e^{-\overline{x}^{2+\eta}/[(2+\eta)^2\widetilde{\mathcal{D}}\beta(t)]}\mathrm{I}_{\overline{\nu}}\left[\frac{2\,(\xi(t)x\overline{x})^{\epsilon}}{(2+\eta)^2\widetilde{\mathcal{D}}\beta(t)}\right], \tag{22}$$

with $\alpha = (1+\eta+\mathcal{K}/\widetilde{\mathcal{D}})/2$ and $\epsilon = (2+\eta)/2$. By using Eq. (22), it is possible to obtain the survival probability and the first passage time distribution. The survival probability, defined as $\mathcal{S}(t) = \int_0^\infty dx\, \rho(x,t)$, is

$$\mathcal{S}(t) = \frac{\overline{x}^{\frac{\overline{\nu}}{2}(2+\eta)} e^{-\overline{x}^{2+\eta}/[(2+\eta)^2\widetilde{\mathcal{D}}\beta(t)]}}{\left((2+\eta)^2\widetilde{\mathcal{D}}\beta(t)\right)^{\overline{\nu}}\Gamma\left(1+\overline{\nu}\right)}\Phi\left[1, 1+\overline{\nu}, \frac{\overline{x}^{2+\eta}}{(2+\eta)^2\widetilde{\mathcal{D}}\beta(t)}\right], \tag{23}$$

where $\Phi(\alpha, \beta, x)$ is the confluent hypergeometric function.[51] The first passage time distribution may be obtained by using the definition presented in Ref. 52, namely, $\mathcal{F}(t) = -\partial\mathcal{S}/\partial t$. For this particular case, it is given by

$$\mathcal{F}(t) = \frac{\overline{x}^{(2+\eta)\overline{\nu}}\dot{\beta}(t)}{\beta(t)\Gamma\left(\overline{\nu}\right)\left[(2+\eta)^2\widetilde{\mathcal{D}}\beta(t)\right]^{\overline{\nu}}} e^{-\overline{x}^{2+\eta}/[(2+\eta)^2\widetilde{\mathcal{D}}\beta(t)]}, \tag{24}$$

with $\dot{\beta}(t) = d\beta(t)/dt$. Figure 3 shows the time behavior of Eq. (24) when a few representative time dependencies for $k(t)$ are considered. Note that, according to the choice of $k(t)$, oscillations are obtained to the survival probability and the first passage time distribution. In Fig. 4, we show the behavior of Eq. (24) for different values of $\mathcal{K}$ in order to illustrate the effect of $\mathcal{K}$. The inset of Fig. 4 illustrates the behavior of the survival probability versus t. It also shows that $\mathcal{K} \neq 0$ increases the value of the survival probability.

III. DISCUSSIONS AND CONCLUSIONS

We have investigated the solutions of Eq. (1) by focusing several situations characterized by a spatial-dependent diffusion coefficient and time-dependent external forces. The first situation analyzed was characterized by the linear external force with an arbitrary time dependence. For this case, we have obtained the exact propagator and analyzed the mean square displacement. The mean square displacement has shown that depending on the choice for the time function in the external force it may present different behaviors and, in particular, the power law dependence which is typical of the anomalous diffusion processes. The subsequent case deal with the exact form of the propagator when a power law dependence was incorporated to the potential. By using the solutions relevant to this situation and restricting the system to a semi-infinite region, i.e., $\rho(0,t) = 0$ and $\rho(\infty, t) = 0$, with the condition $\lim_{t\to\infty}\rho(x,t) = 0$, we have investigated the survival probability and the first passage time distribution. In this framework, we obtain exact expressions for these quantities which, according to the time dependence of the function β, may present a non-usual behavior. This feature is related to the anomalous spreading of the system due to the presence of the spatial dependence in the diffusion coefficient, and the arbitrary time-dependent function presents in the external force. Hopefully, the results found here may be also useful to investigate system where the anomalous diffusion is present.

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