Physica A 388 (2009) 806–810

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Some results for a fractional diffusion equation with radial symmetry in a confined region

E.K. Lenzi [a,*], L.R. da Silva [b], A.T. Silva [a], L.R. Evangelista [a], M.K. Lenzi [c]

[a] Departamento de Física and National Institute of Science and Technology for Complex Systems, Universidade Estadual de Maringá, Avenida Colombo, 5790 - 87020-900 Maringá, Paraná, Brazil

[b] Departamento de Física and National Institute of Science and Technology for Complex Systems, Universidade Federal do Rio Grande do Norte, 59072-970 Natal-RN, Brazil

[c] Departamento de Engenharia Química, Universidade Federal do Paraná, Setor de Tecnologia - Jardim das Américas, Caixa Postal 19011, 81531-990, Curitiba - Paraná, Brazil

ARTICLE INFO

Article history:
Received 7 August 2008
Available online 30 November 2008

PACS:
02.50.-r
05.40.-a
05.90.+m

Keywords:
Fractional diffusion equation
Anomalous diffusion
Green function

ABSTRACT

We investigate an $\mathcal{N}$-dimensional fractional diffusion equation with radial symmetry by taking a spatial and time dependent diffusion coefficient into account, i.e., $\widetilde{\mathcal{D}}(r,t) = \mathcal{D}(t)r^{-\eta}$ with $\mathcal{D}(t) = \mathcal{D}\delta(t) + \overline{\mathcal{D}}(t)$. The equation is considered in a confined region and subjected to time dependent boundary conditions which may be related to inhomogeneous characteristics of the surfaces confining the system. The results show an anomalous spreading of the solutions and an unusual behavior of the survival probability.

1. Introduction

The applications of the fractional diffusion equations [1–6] to anomalous diffusion and the connection to the other formalisms such as continuous random walk formalism [7], Langevin approach [8] and master equation [9] have motivated the study of these equations in order to comprehend their applications to physical contexts where non-conventional dynamical behavior can be found. In this direction, we consider the following fractional diffusion equation

$$\frac{\partial^\gamma}{\partial t^\gamma}\rho(\bar{r},t) = \int_0^t d\bar{t}\;\nabla\cdot\left(\widetilde{\mathcal{D}}(r,t-\bar{t})\nabla\rho(r,\bar{t})\right) \tag{1}$$

by taking the $\mathcal{N}$-dimensional case with radial symmetry into account ($\nabla\cdot(\widetilde{\mathcal{D}}(r,t)\nabla\cdots) \equiv r^{1-\mathcal{N}}\partial_r(r^{\mathcal{N}-1}\widetilde{\mathcal{D}}(r,t)\partial_r\cdots)$), $0 < \gamma \leq 1$ (subdiffusive case), the diffusion coefficient given by $\widetilde{\mathcal{D}}(\bar{r},t) = \mathcal{D}(t)r^{-\eta}$ where $\mathcal{D}(t)$ is an arbitrary time dependent function, and the fractional time derivative considered here is the Caputo derivative [10]. Eq. (1) has as particular cases several situations such as the ones worked out in [11,12] for the two- and three-dimensional cases in a confined region, and the results presented in [13] for cylindrical symmetry. In this manner, many cases presented in the literature are extended to a broad context which may present, for example, different regimes of spreading of the solution or connection to the fractional equation of distributed order for a suitable choice of $\mathcal{D}(t)$.

* Corresponding author. Tel.: +55 04432614330; fax: +55 04432634623.
E-mail address: eklenzi@dfi.uem.br (E.K. Lenzi).

0378-4371/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2008.11.030

To investigate Eq. (1) subjected to time dependent boundary conditions, i.e., inhomogeneous surfaces, we use the Green function approach [14]. The Green function will be useful to find several properties related to Eq. (1) since it contains the dynamical behavior of the system and may clarify the influence of the surface on the spreading of the solution and the behavior of the survival probability. In this respect, the solutions obtained have remarkable characteristics and may find applications, for example, in the adsorption phenomena where the memory effect eventually plays an important role [15,16], in the presence of a reactive boundary [17] and in the first passage time in confined regions [18,19]. Our analysis is first performed by considering Eq. (1) with $\widetilde{\mathcal{D}}(t) = \mathcal{D}\delta(t)r^{-\eta}$, in the absence of reaction term, i.e., $\alpha(r,t) = 0$, subjected to the boundary condition $\rho(r,t)|_{r=a} = \widetilde{\Phi}_a(t)$ and $\rho(r,t)|_{r=b} = \widetilde{\Phi}_b(t)$, where $\widetilde{\Phi}_a(t)$ and $\widetilde{\Phi}_b(t)$ are two arbitrary time dependent functions. The solution for this case has a particular case results found in [11,12,20] and by taking the limit of $a \to 0$ with $\rho(b,t) = 0$ recovers result found in [21]. After, we analyze the solution for Eq. (1) with the diffusion coefficient $\widetilde{\mathcal{D}}(\bar{r}) = (\mathcal{D}\delta(t) + \overline{\mathcal{D}}(t))r^{-\eta}$ and incorporating the reaction term $\alpha(r,t)$ which represents a rate of generation or removal of diffusing substance [22]. These developments are performed in Section 2, and in Section 3 we present our conclusions.

2. Fractional diffusion equation

Let us start our analysis by considering Eq. (1) subjected to the time dependent boundary conditions $\rho(r,t)|_{r=a} = \widetilde{\Phi}_a(t)$ and $\rho(r,t)|_{r=b} = \widetilde{\Phi}_b(t)$, the initial condition $\rho(r,0) = \widetilde{\rho}(r)$ and the diffusion coefficient $\widetilde{\mathcal{D}}(r,t) = \mathcal{D}\delta(t)r^{-\eta}$. For this case, Eq. (1) can be written as follows

$$\frac{\partial^\gamma}{\partial t^\gamma}\rho(r,t) = \mathcal{D}\nabla\cdot\left(r^{-\eta}\nabla\rho(r,t)\right) \tag{2}$$

with $0 < \gamma \le 1$. This equation extends the usual diffusion equation by incorporating the presence of the fractional derivative which produces an anomalous spreading of the initial condition and may be obtained from a continuous time random walk formalism by using the procedure discussed in Ref. [23]. Furthermore, it is noticeable that depending on the choice of $\Phi_a(t)$ and $\Phi_b(t)$ the system may exhibit a stationary solution which is the same as that one obtained from the usual diffusion equation. This feature indicates that the presence of the time fractional derivative produces an anomalous relaxation to a stationary state, in contrast to the spatial fractional derivatives which lead us to the Lévy distributions.

To obtain a solution to Eq. (2) and investigate the influence of the boundary condition on the relaxation of the system and the survival probability, we use the Laplace transform and the Green function approach. After some calculations, it is possible to show that the solution for Eq. (2) in the Laplace space is given by

$$\begin{aligned}\widehat{\rho}(r,s) =& -s^{\gamma-1}\int_a^b \mathrm{d}r' r'^{\mathcal{N}-1}\widehat{\mathcal{G}}(r,r';s)\widetilde{\rho}(r') \\ &+ \left[\mathcal{D}b^{\mathcal{N}-1-\eta}\widetilde{\Phi}_b(s)\frac{\partial}{\partial r'}\widehat{\mathcal{G}}(r,r';s)\bigg|_{r'=b} - \mathcal{D}a^{\mathcal{N}-1-\eta}\widetilde{\Phi}_a(s)\frac{\partial}{\partial r'}\widehat{\mathcal{G}}(r,r';s)\bigg|_{r'=a}\right]\end{aligned} \tag{3}$$

where the last terms correspond to the surface effects and determine the existence of a stationary solution for the process. They may also be connected to several processes, in particular for adsorption on fractals surfaces [24]. The Green function $\widehat{\mathcal{G}}(\bar{r},\bar{r}';s)$ satisfies the equation

$$\mathcal{D}\nabla\cdot\left(r^{-\eta}\nabla\widehat{\mathcal{G}}(r,r';s)\right) - s^\gamma\widehat{\mathcal{G}}(r,r';s) = \frac{1}{r^{\mathcal{N}-1}}\,\delta(r-r') \tag{4}$$

subjected to the conditions $\mathcal{G}(r,r';s)|_{r=b} = 0$ and $\mathcal{G}(r,r';s)|_{r=a} = 0$. Resorting to the eigenfunctions of the Sturm–Liouville problem related to spatial operator of Eq. (4) ($\nabla\cdot\left(r^{-\eta}\nabla\Psi(r,k_n)\right) = -k_n^2\Psi(r,k_n)$ with $\Psi(a,k_n) = \Psi(b,k_n) = 0$), it is possible to show that

$$\widehat{\mathcal{G}}(\bar{r},\bar{r}';s) = -\frac{1}{2+\eta}\sum_{n=1}^{\infty}\frac{\mathcal{N}_n}{s^\gamma + \mathcal{D}k_n^2}\Psi(r,k_n)\Psi(r',k_n) \tag{5}$$

where

$$\Psi(r,k_n) = r^{\frac{1}{2}(2+\eta-\mathcal{N})}\left(\mathrm{J}_\alpha\left(\frac{2k_n}{2+\eta}r^{\frac{1}{2}(2+\eta)}\right)\mathrm{N}_\alpha\left(\frac{2k_n}{2+\eta}a^{\frac{1}{2}(2+\eta)}\right) - \mathrm{J}_\alpha\left(\frac{2k_n}{2+\eta}a^{\frac{1}{2}(2+\eta)}\right)\mathrm{N}_\alpha\left(\frac{2k_n}{2+\eta}r^{\frac{1}{2}(2+\eta)}\right)\right), \tag{6}$$

$\alpha = \mathcal{N}/(2+\eta) - 1$ ($\alpha \ge 0$), $\mathrm{J}_\alpha(x)$ and $\mathrm{N}_\alpha(x)$ are Bessel functions of first and second species, respectively, k_n are solutions of the eigenvalue equation

$$\mathrm{J}_\alpha\left(\frac{2k_n}{2+\eta}b^{\frac{1}{2}(2+\eta)}\right)\mathrm{N}_\alpha\left(\frac{2k_n}{2+\eta}a^{\frac{1}{2}(2+\eta)}\right) - \mathrm{J}_\alpha\left(\frac{2k_n}{2+\theta}a^{\frac{1}{2}(2+\eta)}\right)\mathrm{N}_\alpha\left(\frac{2k_n}{2+\eta}b^{\frac{1}{2}(2+\eta)}\right) = 0, \tag{7}$$

and $\mathcal{N}_n = (\pi k_n)^2/\left[\left(\mathrm{J}_\alpha\left(\frac{2k_n}{2+\eta}a^{\frac{1}{2}(2+\eta)}\right)/\mathrm{J}_\alpha\left(\frac{2k_n}{2+\eta}b^{\frac{1}{2}(2+\eta)}\right)\right)^2 - 1\right]$. By taking the inverse Laplace transform of Eq. (3), we obtain

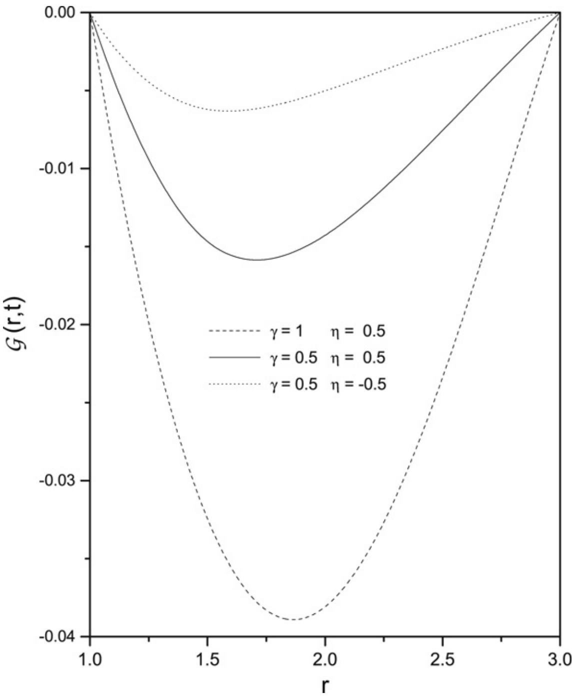

Fig. 1. Behavior of $\mathcal{G}(r,t)$ versus r obtained from Eq. (9) for typical values of γ and η. We consider, for simplicity, $\mathcal{D}=1$, $t=1$, $\mathcal{N}=3$, $a=1$ and $b=3$.

$$\rho(r,t) = -\frac{1}{\Gamma(1-\gamma)}\int_0^t dt' \frac{1}{(t-t')^{\gamma}}\int_a^b dr' r'^{\mathcal{N}-1}\mathcal{G}(r,r';t')\widetilde{\rho}(r') + \int_0^t dt' \left[\mathcal{D}b^{\mathcal{N}-1-\eta}\widetilde{\Phi}_b(t-t')\frac{\partial}{\partial r'}\mathcal{G}(r,r';t')\Big|_{r'=b} - \mathcal{D}a^{\mathcal{N}-1-\eta}\widetilde{\Phi}_a(t-t')\frac{\partial}{\partial r'}\mathcal{G}(r,r';t')\Big|_{r'=a}\right] \quad (8)$$

with

$$\mathcal{G}(r,r';t) = -\frac{t^{\gamma-1}}{2+\eta}\sum_{n=1}^{\infty}\mathcal{N}_n\Psi(r,k_n)\Psi_n(r',k_n)\mathrm{E}_{\gamma,\gamma}\left(-\mathcal{D}k_n^2 t^{\gamma}\right), \quad (9)$$

where $\mathrm{E}_{\alpha,\beta}(x) = \sum_{i=0}^{\infty} x^i/\Gamma(\beta+\alpha i)$ is the generalized Mittag–Leffler function [10] (see Fig. 1).

The presence of this function in the solution indicates an anomalous spreading of the distribution due to the presence of the time fractional derivatives in Eq. (1), which can be verified by analyzing the behavior of the second moment. A noteworthy result may be obtained from Eq. (8) by taking the limit $b \to \infty$, compatible with the boundary condition $\rho(\infty,t)=0$. For this case the solution is given by

$$\rho(r,t) = -\frac{1}{\Gamma(1-\gamma)}\int_0^t dt' \frac{1}{(t-t')^{\gamma}}\int_a^{\infty} dr' r'^{\mathcal{N}-1}\mathcal{G}(r,r';t')\widetilde{\rho}(r') - \mathcal{D}\int_0^t dt'\, a^{\mathcal{N}-1-\eta}\widetilde{\Phi}_a(t-t')\frac{\partial}{\partial r'}\mathcal{G}(r,r';t')\Big|_{r'=a} \quad (10)$$

with the Green function

$$\mathcal{G}(r,r';t) = -\frac{2t^{\gamma-1}}{2+\eta}\int_0^{\infty} dk \frac{k\Psi(r,k)\Psi(r',k)\mathrm{E}_{\gamma,\gamma}\left(-k^2t^{\gamma}\right)}{\mathrm{J}_{\alpha}^2\left(\frac{2k}{2+\eta}a^{\frac{1}{2}(2+\eta)}\right)+\mathrm{N}_{\alpha}^2\left(\frac{2k}{2+\eta}a^{\frac{1}{2}(2+\eta)}\right)}. \quad (11)$$

Now, let us analyze Eq. (1) for the diffusion coefficient $\widetilde{\mathcal{D}}(r,t) = \mathcal{D}(t)r^{-\eta}$, with $\mathcal{D}(t) = \mathcal{D} + \overline{\mathcal{D}}(t)$. We also incorporate in Eq. (1) the reaction or absorption term $\alpha(r,t)$, in order to extend the solution to a broad context and make possible to investigate, for example, drug absorption [25], tumor development [26], and heat flow involving heat production [27]. Applying the same procedure as above, the solution of Eq. (1) for this case is given by

$$\rho(r,t) = -\frac{1}{\Gamma(1-\gamma)}\int_0^t dt' \frac{1}{(t-t')^{\gamma}}\int_a^b dr' r'^{\mathcal{N}-1}\widetilde{\mathcal{G}}(r,r';t')\widetilde{\rho}(r') - \int_0^t dt' \int_a^b dr' r'^{\mathcal{N}-1}\alpha(r',t')\widetilde{\mathcal{G}}(r,r';t-t') + \int_0^t dt' \left[\mathcal{D}b^{\mathcal{N}-1-\eta}\widetilde{\Phi}_b(t-t')\frac{\partial}{\partial r'}\widetilde{\mathcal{G}}(r,r';t')\Big|_{r'=b} - \mathcal{D}a^{\mathcal{N}-1-\eta}\widetilde{\Phi}_a(t-t')\frac{\partial}{\partial r'}\widetilde{\mathcal{G}}(r,r';t')\Big|_{r'=a}\right] \quad (12)$$

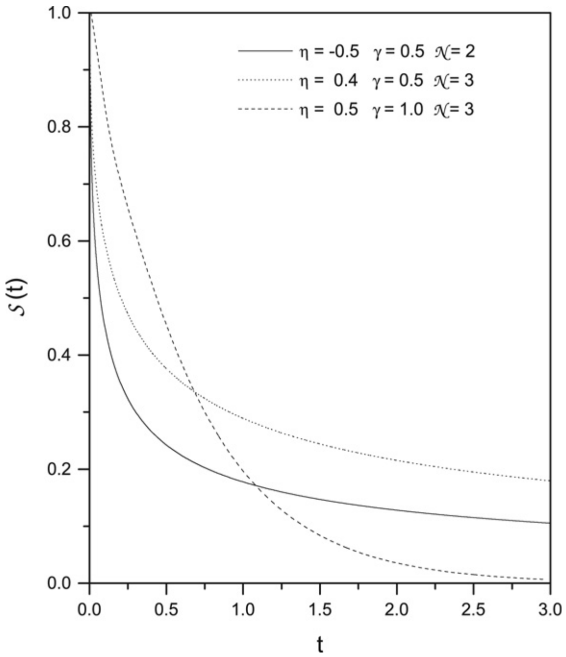

Fig. 2. This figure shows the behavior of $\mathcal{S}(t)$ versus t in order to illustrate Eq. (15) for typical values of $\mathcal{N}$, θ and γ by considering, for simplicity, the boundary condition $\rho(a,t)=0$ and $\rho(b,t)=0$, $\mathcal{D}=1$, $a=1$, $b=3$, and $\rho(\bar{r},0)=\delta(r-3/2)/r^{\mathcal{N}-1}$.

with the Green function given by

$$\widetilde{\mathcal{G}}(r,r';t)=\mathcal{G}(r,r';t)-\frac{1}{2+\eta}\sum_{n=1}^{\infty}\mathcal{N}_n\Psi(r,k_n)\Psi(r',k_n)\Theta(t,k_n) \tag{13}$$

with

$$\Theta(t,k_n)=\sum_{m=1}^{\infty}\frac{(-k_n^2)^m}{m!}\int_0^t \mathrm{d}t_m\overline{\mathcal{D}}(t-t_m)\cdots\int_0^{t_3}\mathrm{d}t_2\overline{\mathcal{D}}(t_3-t_2)\times\int_0^{t_2}\mathrm{d}t_1\overline{\mathcal{D}}(t_2-t_1)t_1^{(m+1)\gamma-1}\mathrm{E}_{\gamma,\gamma}^{(m)}\left(-k_n^2t_1^{\gamma}\right) \tag{14}$$

where $\mathrm{E}_{\alpha,\beta}^{(m)}(y)\equiv d^m\mathrm{E}_{\alpha,\beta}(y)/dy^m$. This result extends results found in Refs. [11–13] by considering the $\mathcal{N}$-dimensional case, a spatial and time dependent diffusion coefficient, and by taking inhomogeneous boundary condition into account. In this direction, results found in Refs. [20,21] were also extended.

By using the above results in the absence of reaction term, we may obtain the survival probability $\mathcal{S}(t)$, which is related to the first passage time distribution, i.e., $\mathcal{F}(t)=-\partial\mathcal{S}(t)/\partial t$ [28]. After some calculations, it is possible to show that it is given by

$$\begin{aligned}\mathcal{S}(t)=&-\frac{a^{\frac{1}{2}(\mathcal{N}-\eta)-1}}{\pi}\sum_{n=1}^{\infty}\frac{\mathcal{N}_n}{k_n^2}\left(1-\frac{b^{\frac{1}{2}(\mathcal{N}-\eta)-1}\mathrm{J}_\alpha\left(\frac{2k_n}{2+\eta}a^{\frac{1}{2}(2+\eta)}\right)}{a^{\frac{1}{2}(\mathcal{N}-\eta)-1}\mathrm{J}_\alpha\left(\frac{2k_n}{2+\eta}b^{\frac{1}{2}(2+\eta)}\right)}\right)\\&\times\left\{\int_a^b\mathrm{d}r'r'^{\mathcal{N}-1}\overline{\rho}(r')\Psi(r,k_n)\Upsilon(t,k_n)-\frac{2+\eta}{\pi^2}\mathcal{D}a^{\frac{1}{2}(\mathcal{N}-\eta)-1}\right.\\&\left.\times\int_0^t\mathrm{d}\bar{t}\;\Theta(t-\bar{t},k_n)\left[\Phi_a(\bar{t})-\frac{b^{\frac{1}{2}(\mathcal{N}-\eta)-1}\mathrm{J}_\alpha\left(\frac{2k_n}{2+\eta}a^{\frac{1}{2}(2+\eta)}\right)}{a^{\frac{1}{2}(\mathcal{N}-\eta)-1}\mathrm{J}_\alpha\left(\frac{2k_n}{2+\eta}b^{\frac{1}{2}(2+\eta)}\right)}\Phi_b(\bar{t})\right]\right\}\end{aligned} \tag{15}$$

with

$$\Upsilon(t,k_n)=\sum_{m=1}^{\infty}\frac{(-k_n^2)^m}{m!}\int_0^t\mathrm{d}t_m\overline{\mathcal{D}}(t-t_m)\cdots\int_0^{t_3}\mathrm{d}t_2\overline{\mathcal{D}}(t_3-t_2)\times\int_0^{t_2}\mathrm{d}t_1\overline{\mathcal{D}}(t_2-t_1)t_1^{m\gamma}\mathrm{E}_{\gamma}^{(m)}\left(-\mathcal{D}k_n^2t_1^{\gamma}\right) \tag{16}$$

(see Fig. 2).

3. Summary and conclusions

We have investigated an $\mathcal{N}$-dimensional fractional diffusion equation by taking the radial symmetry into account. We have also considered spatial and time dependence on the diffusion coefficient, i.e., $\mathcal{D}(r,t) = \mathcal{D}(t)r^{-\eta}$. The fractional diffusion equation was first worked out in the absence of the reaction term with $\mathcal{D}(t) = \mathcal{D}\delta(t)$ by considering an inhomogeneous time dependent boundary condition. The result obtained for this case shows that the distribution has an anomalous spreading due to the presence of the fractional derivative and the time dependence on the boundary condition. It also extends results presented in Refs. [11–13] to a broad context for the subdiffusive case, i.e., $0 < \gamma \leq 1$. Other quantities related to this diffusion process, such as the survival probability, also change. In fact, Eq. (15) shows that for a long time the asymptotic limit is $\mathcal{S}(t) \sim t^{-\gamma}$ when the system is subjected to homogeneous boundary conditions, i.e., $\rho(a,t) = \rho(b,t) = 0$. This behavior is in agreement with the results presented in Ref. [18] for confined geometry. Afterwards, we have studied the changes produced by incorporating a time dependence on the diffusion coefficient, i.e., $\mathcal{D}(t) = \mathcal{D} + \overline{\mathcal{D}}(t)$ and the reaction term $\alpha(r,t)$ on the solution. This time dependence on the solution may produce different diffusive regimes on the solution and may be related to the fractional diffusion equations of distributed order. Finally, we expect that the results found here may be useful to study the systems where the anomalous diffusion is present and the surface plays an important role.

Acknowledgments

We thank CNPq, CAPES and Fundação Araucária for partial financial support.

References

[1] R. Hilfer, R. Metzler, A. Blumen, J. Klafter, Strange kinetics, Chem. Phys. 284 (1–2) (2004).
[2] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[3] R. Metzler, J. Klafter, Phys. Rep. 339 (2000) 1.
[4] G.M. Zaslavsky, Phys. Rep. 371 (2002) 461.
[5] W.R. Schneider, W. Wyss, J. Math. Phys. 30 (1989) 134.
[6] B.N.N. Achar, J.W. Hanneken, J. Mol. Liq. 114 (2004) 147.
[7] W.T. Coffey, D.S.F. Crothers, D. Holland, S.V. Titov, J. Mol. Liq. 114 (2004) 165.
[8] S. Eule, R. Friedrich, F. Jenko, D. Kleinhans, J. Phys. Chem. B 111 (2007) 11474.
[9] R. Metzler, E. Barkai, J. Klafter, Europhys. Lett 46 (1999) 431.
[10] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[11] R. Rossato, M.K. Lenzi, L.R. Evangelista, E.K. Lenzi, Phys. Rev. E 76 (2007) 032102.
[12] L.S. Lucena, L.R. da Silva, L.R. Evangelista, M.K. Lenzi, R. Rossato, E.K. Lenzi, Chem. Phys. 344 (2008) 90.
[13] Y.Z. Povstenko, J. Mol. Liq. 137 (2008) 137.
[14] M.P. Morse, H. Feshbach, Methods of Theoretical Physics, McGrawHill, New York, 1953.
[15] G. Barbero, L.R. Evangelista, Adsorption Phenomena and Anchoring Energy in Nematic Liquid Crystals, Taylor & Francis, London, 2006.
[16] R.S. Zola, E.K. Lenzi, L.R. Evangelista, G. Barbero, Phys. Rev. E 75 (2007) 0422601.
[17] M.A. Lomholt, I.M. Zaid, R. Metzler, Phys Rev. Lett. 98 (2007) 200603.
[18] S. Condamin, O. Bénichou, J. Klafter, Phys. Rev. Lett. 98 (2007) 250602.
[19] S. Condamin, O. Bénichou, V. Tejedor, R. Voituriez, J. Klafter, Nature 450 (2007) 77.
[20] T. Bickel, Physica A 377 (2007) 24.
[21] E.K. Lenzi, R.S. Mendes, J.S. Andrade, L.R. da Silva, L.S. Lucena, Phys. Rev. E 71 (2005) 052101.
[22] J. Crank, The Mathematics of Diffusion, Oxford University Press, London, 1956.
[23] B.I. Henry, T.A.M. Langlands, S.L. Wearne, Phys. Rev. E 74 (2006) 031116.
[24] M. Giona, M. Giustiniani, J. Phys. Chem. 100 (1996) 16690.
[25] P. Macheras, A. Iliadis, Modeling in Biopharmaceutics, Pharmacokinetics, and Pharmacodynamics, Springer, New York, 2006.
[26] A. Iomin, J. Phys.: Conf. Ser. 7 (2005) 57.
[27] H.S. Carslaw, J.C. Jaeger, Conduction of Heat in Solids, Oxford University Press, London, 1959.
[28] S. Redner, A Guide to First-Passage Processes, Cambridge University Press, Cambridge, England, 2001.