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Solutions for a fractional nonlinear diffusion equation with external force and absorbent term

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Abstract. We devote this work to investigate the solutions of a $\mathcal{N}$-dimensional nonlinear fractional diffusion equation which emerges from the continuity equation by considering a nonlinear fractional generalization of Darcy law and incorporating an absorbent term. The solutions obtained show a nonusual spreading of the distribution and a compact or long tail behavior which can be related to the anomalous diffusion.

Keywords: exact results, transport processes/heat transfer (theory), diffusion

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Contents

1. Introduction

The broadness of physical situations related to anomalous diffusion has attracted the attention of many researchers [1]–[6]. In fact, it is present in several contexts such as diffusion on fractals [7, 8], relaxation to equilibrium in systems (e.g. polymers chains and membranes) with long temporal memory [9], tumor development [10], transport of fluid in porous media [11], flow through porous media [11], stock market fluctuation [12], behavior of the heartbeat [13], amorphous semiconductors [14] and micelles present in salt water [15]. In these situations, where anomalous diffusion is present, the second moment can be either finite [16] or not [16, 17]. If the second moment, $\langle r^2 \rangle$, is finite, it is generally given by $\langle r^2 \rangle \sim t^{\alpha}$, where $\alpha < 1$ corresponds to a subdiffusion process, $\alpha = 1$ is the usual diffusion process and $\alpha > 1$ indicates a superdiffusion process. For these cases, the diffusion phenomena may be analyzed by equations which employ fractional time derivatives [2, 9, 18] ($\partial_t \rho = \mathcal{D}_\gamma \, {}_0D_t^{1-\gamma}(\nabla^2 \rho)$), nonlinear terms such as the porous media ($\partial_t \rho = \mathcal{D}_\nu \nabla^2 \rho^\nu$) or a generalized diffusion equations which consider a spatial-and time-dependent diffusion coefficient [19] ($\partial_t \rho = \nabla \cdot (\mathcal{D}(r,t) \nabla \rho)$). These equations have been successfully used to describe several physical situations. In fact, the fractional time diffusion equation has been applied to transport phenomena in disordered media [20], non-Markovian dynamic processes [21], transport of chemicals through membranes [22], comb-like models, axial transport of granular materials [23] and asymmetric translocation of DNA molecules [24]. The porous media equation may be found in applications in percolation of gases through porous media [25], thin saturated regions in porous media [26], thin liquid films spreading under gravity [27] and a solid-on-solid model for surface growth [28]. The usual diffusion equation with a spatial time-dependent diffusion coefficient has been used to investigate turbulent processes (Richardson law [29] and Kolmogorov law [30]), diffusion on fractals [31], fast electrons in a hot plasma in the presence of a electric field [32] and hydrologic systems [33]. On the other hand, anomalous diffusion processes whose second moment is not finite are essentially characterized by Lévy distributions [17]. This context may be investigated, for example, by the diffusion equations which present fractional derivatives applied to spatial variable ($\partial_t \rho = \mathcal{D}_\mu \nabla^\mu \rho$) [2, 17] or nonlinear terms ($\partial_t \rho = \mathcal{D}_\mu \nabla^\mu \rho^\nu$). The solution of these diffusion equations is given in terms of Lévy distributions or solutions whose asymptotic behavior is long tailed as a Lévy distribution.

As observed from the above discussion, anomalous diffusion can be investigated by different types of equations. Therefore, the study of these diffusion equations represents an important tool towards a comprehension of the physical phenomena whose diffusion processes are not the usual ones. In this direction, this paper reports an investigation of time-dependent solutions of an $\mathcal{N}$-dimensional nonlinear fractional diffusion equation which interpolates the diffusion equations with nonlinear terms and spatial fractional derivatives. More precisely, we investigate the following equation:

$$\frac{\partial}{\partial t}\rho(r,t) = \frac{1}{r^{\mathcal{N}-1}}\frac{\partial}{\partial r}\left\{r^{\mathcal{N}-1}\left[\widetilde{\mathcal{D}}(r,t)\left|\frac{\partial^{\mu'}}{\partial r^{\mu'}}[\rho(r,t)]^{\delta}\right|^{n}\frac{\partial^{\mu}}{\partial r^{\mu}}\left[\rho(r,t)\right]^{\nu}\right]\right\} - \frac{1}{r^{\mathcal{N}-1}}\frac{\partial}{\partial r}\left[r^{\mathcal{N}-1}F(r,t;\rho)\rho(r,t)\right] - \alpha(t)[\rho(r,t)]^{\overline{\mu}}, \qquad (1)$$

where $\widetilde{\mathcal{D}}(r,t)$ is a spatial-and time-dependent diffusion coefficient, $F(r,t;\rho)$ represents an external force applied to the system and the last term of the equation is related to a reaction expression. The fractional operators ($\partial_r^{\mu'}$ and ∂_r^{μ}) applied to the spatial variable are the Riemann–Liouville operators [34]. Equation (1) may be obtained, from the mathematical point of view, employing the procedure presented in [35] with suitable changes in order to incorporate the reaction term and the fractional derivatives. In fact, by incorporating the nonlinear reaction term $\alpha(t)[\rho(r,t)]^{\overline{\mu}}$ as absorbent in the equation of continuity ($\partial_t\rho + \nabla\cdot\mathcal{J} = 0$) and taking the generalized Darcy law $\mathcal{J}(r,t) = -\widetilde{\mathcal{D}}(r,t)|\partial_r^{\mu'}[\rho(r,t)]^{\delta}|^{n}\partial_r^{\mu}[\rho(r,t)]^{\nu} + F(r,t;\rho)\rho(r,t)$ into account it may be obtained. Notice that the previous mathematical construction, used to incorporate the reaction term and the spatial fractional derivatives, did not explicitly deal with the long jumps which are present in the fractional diffusion equations. This discussion about the long jump and fractional diffusion equations with reaction terms, which is a hard task, can be found in [36]–[38]. For this reason, and to incorporate nonlinear aspects, we restrict our discussion to the above procedure to formulate a nonlinear fractional diffusion equation which interpolates several situations. The diffusion coefficient analyzed here is given by $\widetilde{\mathcal{D}}(r,t) = \mathcal{D}(t)r^{-\theta}$ with an arbitrary time dependence on $\mathcal{D}(t)$. The spatial dependence present in the diffusion coefficient has been employed to investigate several physical contexts, for example, diffusion on fractals [7, 8], and motivated the theoretical study of the usual diffusion equation in a broad context. The external force $F(r,t;\rho)$ considered here depends on the distribution $\rho(r,t)$ of the system and is given by $F(r,t;\rho) = -k(t)r - \mathcal{K}r^{\gamma}[\rho(r,t)]^{\overline{\eta}-1}$. This external force incorporated in equation (1) leads us to an extension of some well-known cases such as the Ornstein–Uhlenbeck process, Rayleigh process [19] and Burgers' equation [39]. Other particular cases of the above equation can be found in [40]–[44]. In addition, the potential related to this external force has, as a particular case, the logarithmic potential used, for instance, to establish the connection between the fractional diffusion coefficient and the generalized mobility [45]. The presence of the reaction term is particularly interesting in heterogeneous catalytic processes, disordered systems [46]–[48], species coagulation ($A + A \to 0$ or $mA \to lA$ ($m > l$)) and irreversible first-order reactions [49]. It also appears when a tracer undergoing radioactive decay is transported through a porous medium and in the heat flow involving heat production [50]. For $\alpha(t) = 0$, it can be verified for equation (1) that $\int_{-\infty}^{\infty} \mathrm{d}r\, r^{\mathcal{N}-1}\rho(r,t)$ is time-independent (hence, if ρ is normalized at $t = 0$, it will remain so for ever).

The plan of this work is to investigate for several situations the time-dependent solutions of equation (1). In section 2, we start our analysis by considering equation (1) subjected to a linear absorbent term, i.e. $\overline{\mu} = 1$, and external forces. For this case, we first study the situation characterized by the absence of external forces and $\mu' = \mu = 1$. Subsequently, we incorporate the nonlinear external force $F(r,t;\rho) = -k(t)r - \mathcal{K}r^{\gamma}[\rho(r,t)]^{\overline{\eta}-1}$ and discuss the changes produced on solution by the linear term and the nonlinear term. Following this, we analyze equation (1) by considering a nonlinear absorbent term, i.e. $\mu \neq 1$. After these developments, we discuss the case $\mu \neq 1$ and $\mu' \neq 1$ without external forces and absorbent terms. A connection with the Tsallis formalism is also discussed for these cases. In section 3, we present our conclusions and discussions.

2. Nonlinear fractional diffusion equation

We start our investigation of the time-dependent solutions of the nonlinear fractional diffusion equation (1) with the case $\overline{\mu} = \mu' = \mu = 1$, i.e., without fractional derivatives and with a linear reaction term. Afterwards, we analyze the case characterized by a nonlinear reaction term, i.e. $\overline{\mu} \neq 1$, and the presence of the fractional derivatives in the diffusive term, i.e. $\mu' \neq 1$, $\mu \neq 1$. The solutions are obtained by using the similarity method, which makes it possible to reduce this partial differential equation to ordinary differential equations. The form of these ordinary differential equations depend on the boundary conditions or the restrictions in the form of conservation laws. In this manner, we restrict our investigation to solutions of the type

$$\overline{\rho}(r,t) = \left(\frac{1}{\Phi(t)}\right)^{\mathcal{N}} \tilde{\rho}\left(\frac{r}{\Phi(t)}\right). \tag{2}$$

The solutions obtained from this procedure satisfy the boundary condition $\rho(\infty,t) = 0$ and the normalization when $\overline{\alpha}(t) = 0$. Following this, we consider the distribution which satisfies equation (1) by accounting for the above conditions on the parameters $\overline{\mu}$, μ' and μ, given by $\rho(r,t) = \mathrm{e}^{-\int_0^t \mathrm{d}\tilde{t}\overline{\alpha}(\tilde{t})}\overline{\rho}(r,t)$, where $\overline{\rho}(r,t)$ is a function to be obtained and is formally given by equation (2). By substituting $\rho(r,t)$ in equation (1) and performing some calculations, we obtain

$$\frac{\partial}{\partial t}\overline{\rho}(r,t) = \frac{\overline{\mathcal{D}}(t)}{r^{\mathcal{N}-1}}\frac{\partial}{\partial r}\left\{r^{\mathcal{N}-1}\left[r^{-\theta}\left|\frac{\partial}{\partial r}[\overline{\rho}(r,t)]^{\delta}\right|^{n}\frac{\partial}{\partial r}[\overline{\rho}(r,t)]^{\nu}\right]\right\} - \frac{\partial}{\partial r}[F(r,t;\rho)\overline{\rho}(r,t)] \tag{3}$$

with $\overline{\mathcal{D}}(t) = \mathcal{D}(t)\exp[-(\nu + n\delta - 1)\int_0^t \mathrm{d}\tilde{t}\,\overline{\alpha}(\tilde{t})]$. From equation (3), we analyze the solutions by considering the following situations: (i) the absence of external forces, (ii) the presence of a linear external force, i.e. $F(r,t) = -k(t)r$ and (iii) the external force $F(r,t) = -k(t)r - \mathcal{K}r^{\gamma}[\rho(r,t)]^{\overline{\eta}-1}$ accomplished by $\mathcal{D}(t) = \mathcal{D} = \mathrm{const}$ and $\alpha(t) = 0$.

For the first situation, i.e. the absence of external force, equation (3) reduces to

$$\frac{\partial}{\partial t}\overline{\rho}(r,t) = \frac{\overline{\mathcal{D}}(t)}{r^{\mathcal{N}-1}}\frac{\partial}{\partial r}\left\{r^{\mathcal{N}-1}\left[r^{-\theta}\left|\frac{\partial}{\partial r}[\overline{\rho}(r,t)]^{\delta}\right|^{n}\frac{\partial}{\partial r}[\overline{\rho}(r,t)]^{\nu}\right]\right\}. \tag{4}$$

This equation may also be formulated in the context of heat diffusion in order to describe an anomalous process of the spreading of the temperature. In this scenario, equation (4) is obtained by using the thermal conductivity dependent on the temperature [51]. By applying the similarity method to equation (4), i.e. considering that the solution is given by equation (2), we may reduce the above diffusion equation to two ordinary differential equations using a constant of separation $\overline{k}$, determined by the normalization condition. After some calculations, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{z^{\mathcal{N}-1}\left[z^{-\theta}\left|\frac{\mathrm{d}}{\mathrm{d}z}[\tilde{\rho}(z)]^{\delta}\right|^{n}\frac{\mathrm{d}}{\mathrm{d}z}[\tilde{\rho}(z)]^{\nu}\right]\right\}=-\overline{k}\frac{\mathrm{d}}{\mathrm{d}z}[z^{\mathcal{N}}\tilde{\rho}(z)] \tag{5}$$

and

$$[\Phi(t)]^{\xi-2}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t)=\overline{k}\;\overline{\mathcal{D}}(t) \tag{6}$$

with $\xi=3+n+\theta+(\nu+\delta n-1)\mathcal{N}$ and $z=r/\Phi(t)$. Solving equation (6), we find that

$$\frac{\Phi(t)}{\Phi(0)}=\left[1+k'\int_0^t \mathrm{d}\tilde{t}\,\overline{\mathcal{D}}(\tilde{t})\right]^{1/(\xi-1)} \tag{7}$$

where $k'=(\xi-1)\overline{k}/(\Phi(0))^{\xi-1}$ and the presence of $\Phi(0)$ may be related to a distribution which has an initial shape. Note also that similar time-dependent behavior has been found in other nonlinear diffusion equations [40]–[44]. This fact indicates that different diffusion equations may have similar anomalous spreading for the probability distribution. An integration on equation (5) yields

$$z^{-\theta}\left|\frac{\mathrm{d}}{\mathrm{d}z}[\tilde{\rho}(z)]^{\delta}\right|^{n}\frac{\mathrm{d}}{\mathrm{d}z}[\tilde{\rho}(z)]^{\nu}=-\overline{k}z\tilde{\rho}(z). \tag{8}$$

To find the solution of equation (8), we use the ansatz: $\tilde{\rho}(z)=(1-\alpha z^{\lambda})^{\beta}$, which satisfies the boundary condition $\rho(\infty,t)=0$, with the parameters β, α and λ to be determined. By substituting the ansatz in equation (8), we obtain

$$\beta=\frac{n+1}{\nu+\delta n-1};\qquad \lambda=\frac{2+\theta+n}{n+1};\qquad \alpha=\frac{\nu+\delta n-1}{2+\theta+n}\left(\frac{\overline{k}}{\delta^{n}\nu}\right)^{1/(n+1)}. \tag{9}$$

Thus, $\tilde{\rho}(z)$ can be written as follows:

$$\tilde{\rho}(z)=\left(1-\frac{\nu+\delta n-1}{2+\theta+n}\left(\frac{\overline{k}}{\delta^{n}\nu}\right)^{1/(n+1)}z^{(2+\theta+n)/(n+1)}\right)^{(n+1)/(\nu+\delta n-1)} \tag{10}$$

and from this result $\overline{\rho}(r,t)$ can be found (see figure 1). At this point, it is interesting to note that equation (10) may be expressed in terms of the q-exponential function which is present in the Tsallis formalism [52]. This function is defined as $\exp_q[r]=[1+(1-q)r]^{1/1-q}$ for $1+(1-q)r\geq 0$ and $\exp_q[r]=0$ for $1+(1-q)r\leq 0$. By choosing $q=2-(\nu+\delta n)$, we can identify the structure present in equation (10) with a q-exponential as follows:

$$\widetilde{\rho}(z)=\exp_q^{1+n}\left[-\frac{1}{2+\theta+n}\left(\frac{\overline{k}}{\delta^{n}\nu}\right)^{1/(n+1)}z^{(2+\theta+n)/(n+1)}\right]. \tag{11}$$

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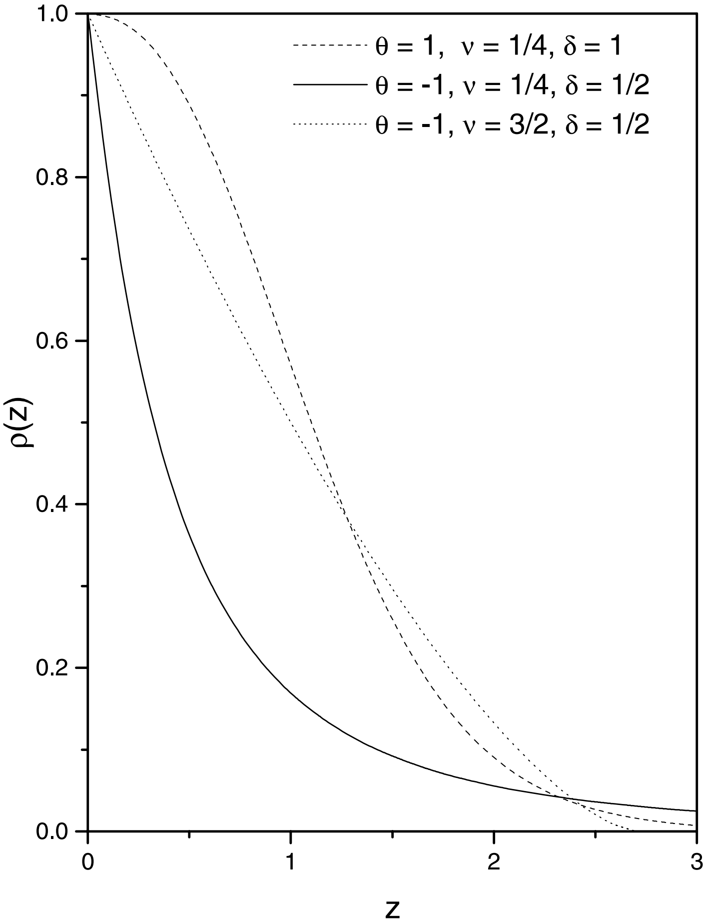

Figure 1. Behavior of $\rho(z)$ versus z obtained from equation (10). We consider, for simplicity, $\mathcal{D} = 1$, $n = 1/2$ and $\mathcal{N} = 3$.

This connection, between equation (10) and the Tsallis formalism, suggests a thermostatistical context different from the usual one for this equation and indicates that the solution can have a compact or a long tailed behavior as a Lévy distribution depending on the values of the parameters ν, n and θ. For the case $\alpha(t) = 0$, the anomalous spreading of the distribution can be analyzed by considering the second moment of the distribution (11) which, for this case, is $\langle r^2 \rangle \propto t^{2/(\xi-1)}$. This result for the second moment shows that, for $2/(\xi - 1) < 1, = 1, > 1$, we may have a sub-, normal or superdiffusive behavior depending on the values of the parameters ν, θ, n and $\mathcal{N}$. Another remarkable feature concerning equation (11) is the asymptotic connection with the Lévy distribution for large z since $\rho(z) \sim 1/z^{3+\mu}$ with $\mu = (5 + n + \theta - 3q)/(q - 1)$.

Now, we analyze the changes produced by the external forces in the solution obtained above. In this direction, we first incorporate a linear external force in our analysis, i.e. $F(r, t) = -k(t)r$. A similar external force may be related to the Ornstein–Uhlenbeck process. Thus, the result found for equation (1), with this external force, extends this process to a nonlinear case. In this direction, a Langevin equation may be found by using the procedure developed in [53]. Afterwards, we investigate a situation characterized by the external force $F(r, t) = -k(t)r - \mathcal{K}r^{\gamma}[\rho(r, t)]^{\overline{\eta}-1}$. By taking this linear external force into account, equation (1) can be written as follows:

$$\frac{\partial}{\partial t}\overline{\rho}(r,t) = \frac{1}{r^{\mathcal{N}-1}}\frac{\partial}{\partial r}\left\{r^{\mathcal{N}-1}\left[\mathcal{D}(t)r^{-\theta}\left|\frac{\partial}{\partial r}[\overline{\rho}(r,t)]^{\delta}\right|^{n}\frac{\partial}{\partial x}[\overline{\rho}(r,t)]^{\nu}\right]\right\} + \frac{k_1(t)}{r^{\mathcal{N}-1}}\frac{\partial}{\partial r}[r^{\mathcal{N}}\overline{\rho}(r,t)]. \quad (12)$$

In order to investigate the solutions which emerge from the above equation, we use the procedure employed before for the case without external forces. By substituting equation (2) in equation (12), we obtain

$$-\frac{\dot{\Phi}(t)}{[\Phi(t)]^2}\frac{\mathrm{d}}{\mathrm{d}\overline{z}}[\overline{z}^{\mathcal{N}}\tilde{\rho}(\overline{z})] = \frac{\overline{\mathcal{D}}(t)}{[\Phi(t)]^{\xi}}\frac{\mathrm{d}}{\mathrm{d}\overline{z}}\left\{z^{-\theta}\left|\frac{\mathrm{d}}{\mathrm{d}\overline{z}}[\tilde{\rho}(\overline{z})]^{\delta}\right|^{n}\frac{\mathrm{d}}{\mathrm{d}\overline{z}}[\tilde{\rho}(\overline{z})]^{\nu}\right\} + \frac{k_1(t)}{\Phi(t)}\frac{\mathrm{d}}{\mathrm{d}\overline{z}}[\overline{z}^{\mathcal{N}}\tilde{\rho}(\overline{z})] \quad (13)$$

with $\overline{z} = r/\Phi(t)$, as for the previous case. By introducing a constant of separation $\overline{k}$, we may separate the above equation into two equations. One of them depends on the z variable and another one only involves a temporal variable, as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\overline{z}}\left\{\overline{z}^{\mathcal{N}-1-\theta}\left|\frac{\mathrm{d}}{\mathrm{d}\overline{z}}[\tilde{\rho}(\overline{z})]^{\delta}\right|^{n}\frac{\mathrm{d}}{\mathrm{d}\overline{z}}[\tilde{\rho}(\overline{z})]^{\nu}\right\} = -\overline{k}\frac{\mathrm{d}}{\mathrm{d}\overline{z}}[\overline{z}^{\mathcal{N}}\tilde{\rho}(z)] \quad (14)$$

and

$$[\Phi(t)]^{\xi-2}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) + k_1(t)[\Phi(t)]^{\xi-1} = \overline{k}\,\overline{\mathcal{D}}(t). \quad (15)$$

From these equations, one can observe that the presence of a linear external force in equation (12) only modifies the time-dependent behavior of the distribution. Equation (14) was solved before, so our focus will be on equation (15). After some calculations, it can be shown that the solution of equation (15) is given by

$$\frac{\Phi(t)}{\Phi(0)} = \left[1 + k'\int_0^t \mathrm{d}\tilde{z}\,\overline{\mathcal{D}}(\tilde{t})\exp\left((\xi-1)\int_0^{\tilde{t}} k_1(t')\mathrm{d}t'\right)\right]^{1/(\xi-1)}\exp\left(-\int_0^t k_1(t')\mathrm{d}t'\right) \quad (16)$$

(see figure 2). This result shows that, depending on the choice of the time dependence of the external force, one can obtain a second moment that is reduced to a constant when long times are considered. The importance of this fact is the feasibility of a steady-state solution, which is not observed for the situation characterized by the absence of external forces.

Following our analysis, we incorporate the term $-\mathcal{K}r^{\gamma}[\rho(r,t)]^{\overline{\eta}-1}$ in the previous external force, leading us to $F(r,t;\rho) = -k(t)r - \mathcal{K}r^{\gamma}[\rho(r,t)]^{\overline{\eta}-1}$ and employ $\mathcal{D}(t) = \mathcal{D} = \text{const}$ with $\alpha(t) = 0$. Solving the nonlinear diffusion equation with this external force is a hard task: however, since we are interested in the solution which can be expressed in terms of scaled functions as equation (2), we restrict the values of the parameters γ and η to ones that verify the following equation: $\mathcal{N}\overline{\eta} - \gamma = \xi + \mathcal{N} - 2$. By taking these conditions into account for this external force and employing the above procedure, it is possible to show that equation (1) may be simplified to the following equations:

$$-\overline{k}\frac{\mathrm{d}}{\mathrm{d}z}[z^{\mathcal{N}}\tilde{\rho}(z)] = \mathcal{D}\frac{\mathrm{d}}{\mathrm{d}z}\left\{z^{\mathcal{N}-1}\left[z^{-\theta}\left|\frac{\mathrm{d}}{\mathrm{d}z}[\tilde{\rho}(z)]^{\delta}\right|^{n}\frac{\mathrm{d}}{\mathrm{d}z}[\tilde{\rho}(z)]^{\nu}\right]\right\} + \mathcal{K}\frac{\mathrm{d}}{\mathrm{d}z}\left[z^{\mathcal{N}-1}\left(z^{\gamma}[\tilde{\rho}(z)]^{\overline{\eta}}\right)\right] \quad (17)$$

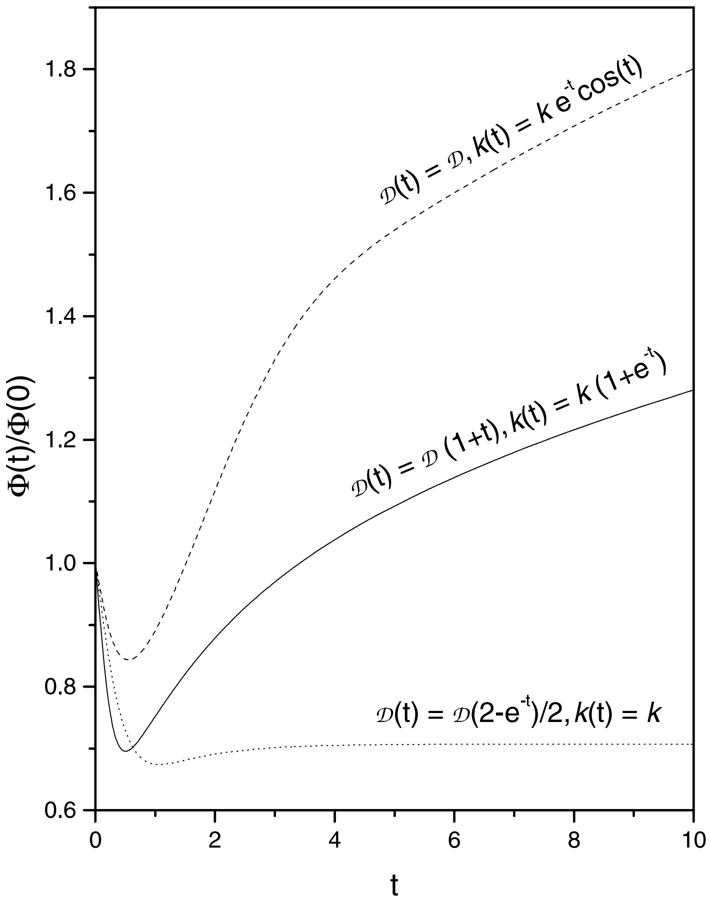

Figure 2. This figure shows the behavior of $\Phi(t)$ versus t in order to illustrate the behavior of equation (16). We consider, for simplicity, $\Phi(0) = 1$, $\mathcal{D} = 1/((\xi-1)\overline{k})$, $\xi = 5$ and $\overline{k} = 1$.

and

$$[\Phi(t)]^{\xi-2}\frac{\mathrm{d}}{\mathrm{d}z}\Phi(t) + k_1(t)[\Phi(t)]^{\xi-1} = \overline{k}. \tag{18}$$

The solution of equation (18) is given by

$$\frac{\Phi(t)}{\Phi(0)} = \left[1 + \widetilde{k}\int_0^t \mathrm{d}\tilde{t}\exp\left((\xi-1)\int_0^{\tilde{t}}\mathrm{d}t'\,k_1(t')\right)\right]^{1/(\xi-1)}\exp\left(-\int_0^t \mathrm{d}t'\,k_1(t')\right) \tag{19}$$

where $\widetilde{k} = (\xi-1)\overline{k}/(\Phi(0))^{\xi-1}$. Notice that the result obtained above for equation (18) is essentially the same as equation (16) obtained for equation (15). This feature implies that the term $-\mathcal{K}r^{\gamma}[\rho(r,t)]^{\overline{\eta}-1}$ present in the external force only changes the spatial behavior of the solution and not its spreading when subjected to the above conditions. In order to obtain the solution of equation (17), we first perform an integration yielding

$$\mathcal{D}z^{-\theta}\left|\frac{\mathrm{d}}{\mathrm{d}z}[\tilde{\rho}(z)]^{\delta}\right|^{n}\frac{\mathrm{d}}{\mathrm{d}z}[\tilde{\rho}(z)]^{\nu} = -(\overline{k}z + \mathcal{K}z^{\gamma}[\tilde{\rho}(z)]^{\overline{\eta}-1})\tilde{\rho}(z). \tag{20}$$

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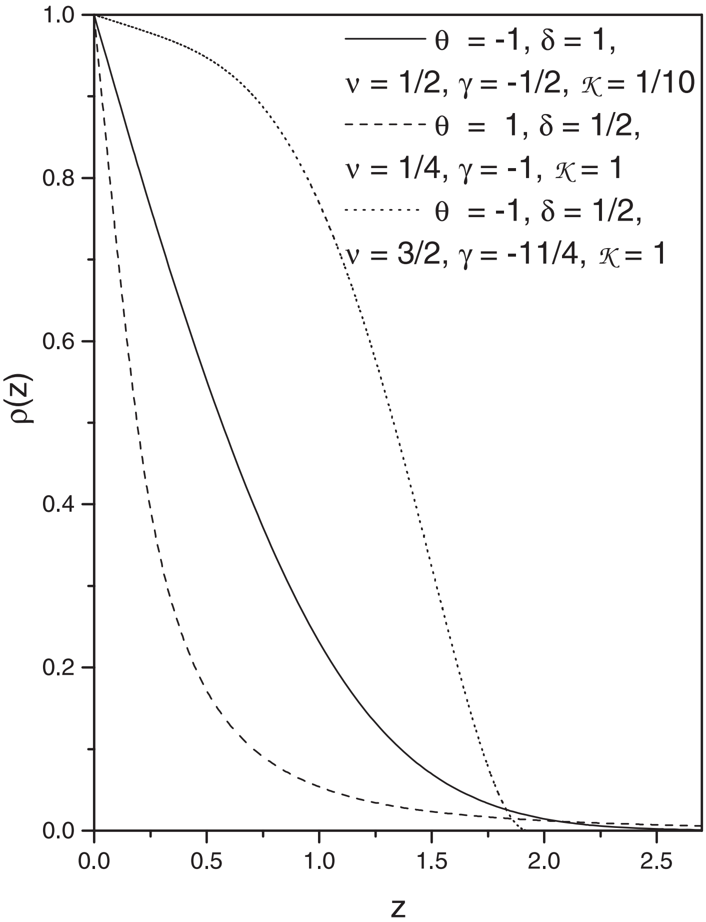

Figure 3. Behavior of $\rho(z)$ versus z obtained from equation (22). We consider, for simplicity, $\mathcal{D} = 1$, $n = 1/2$ and $\mathcal{N} = 3$.

The constant of integration was eliminated due to the boundary condition $\rho(\infty, t) = 0$. The solution of equation (20) is formally given by

$$\tilde{\rho}(z) = \exp_q^{1+n}\left[-\frac{1}{n+1}\left(\frac{\overline{k}}{\mathcal{D}\delta^n\nu}\right)^{1/(n+1)}\int^z \mathrm{d}\overline{z}\left(\overline{z}^{1+\theta} + \frac{\mathcal{K}}{\overline{k}}\overline{z}^{\gamma+\theta}[\tilde{\rho}(\overline{z})]^{\overline{\eta}-1}\right)^{1/(n+1)}\right]. \quad (21)$$

This solution is implicit and extends the result found in [40, 42]. For the special case $\overline{\eta} = 1$, equation (21) makes an explicit solution on the variable z, which can be written as follows:

$$\tilde{\rho}(z) = \exp_q^{1+n}\left[-\frac{1}{n+1}\left(\frac{\overline{k}}{\mathcal{D}\delta^n\nu}\right)^{1/(n+1)}\int^z \mathrm{d}\overline{z}\left(\overline{z}^{1+\theta} + \frac{\mathcal{K}}{\overline{k}}\overline{z}^{\gamma+\theta}\right)^{1/(n+1)}\right] \quad (22)$$

(see figure 3) and for $n = 0$ the result found in [16] is formally recovered after performing the integration.

Let us investigate equation (1) by considering the case $\overline{\mu} \neq 1$, with $\mathcal{D}(t) = \mathcal{D} = \text{const}$, $\overline{\alpha}(t) = \overline{\alpha} = \text{const}$ and, for simplicity, $F(r, t) = \mathcal{K}r^{\gamma}[\rho(r, t)]^{\overline{\eta}-1}$. For this case, equation (1) is given by

$$\frac{\partial}{\partial t}\rho(r,t)=\frac{\mathcal{D}}{r^{\mathcal{N}-1}}\frac{\partial}{\partial r}\left\{r^{\mathcal{N}-1}\left[r^{-\theta}\left|\frac{\partial}{\partial r}[\rho(r,t)]^{\delta}\right|^{n}\frac{\partial}{\partial r}\left[\rho(r,t)\right]^{\nu}\right]\right\} -\frac{1}{r^{\mathcal{N}-1}}\frac{\partial}{\partial r}\left[r^{\mathcal{N}-1}\left(r^{\gamma}[\rho(r,t)]^{\overline{\eta}}\right)\right]+\overline{\alpha}[\rho(r,t)]^{\mu}. \tag{23}$$

Following the approach employed in [40], we consider that the solution of equation (23) is given by

$$\rho(r,t)=\phi(t)\mathcal{P}(\zeta(t)r) \tag{24}$$

where $\phi(t)$ and $\zeta(t)$ are time-dependent functions to be found. In this direction, it is interesting initially to analyze the kinetic equation which emerges from equation (23) for $\mathcal{D}=0$, i.e. the equation $\partial_t\rho(t)=-\alpha[\rho(t)]^{\overline{\mu}}$. The solution for this kinetic equation is known and given by $\rho(t)\propto[1-(1-\overline{\mu})\alpha t]^{1/(1-\overline{\mu})}$. This result suggests we employ $\phi(t)=[1-(1-\overline{\mu})kt]^{1/(1-\overline{\mu})}$ which recovers the previous solution for the case $\mathcal{D}=0$. To obtain $\zeta(t)$, we may substitute the solution proposed for $\rho(r,t)$, equation (24), into equation (23) with $\phi(t)$ given above. These considerations yield

$$\zeta(t)=[1-(1-\overline{\mu})kt]^{(n+\nu-\overline{\mu})((1-\overline{\mu})(2+n+\theta))} \tag{25}$$

and

$$-k\mathcal{P}(\eta)-\left(\frac{\mu-\delta n-\nu}{2+\theta+n}\right)k\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\mathcal{P}(\eta)=\frac{\mathcal{D}}{\eta^{\mathcal{N}-1}}\frac{\mathrm{d}}{\mathrm{d}\eta}\left\{\eta^{\mathcal{N}-1}\left[\eta^{-\theta}\left|\frac{\mathrm{d}}{\mathrm{d}\eta}\mathcal{P}(\eta)\right|^{n}\frac{\mathrm{d}}{\mathrm{d}\eta}[\mathcal{P}(\eta)]^{\nu}\right]\right\} +\frac{\mathrm{d}}{\mathrm{d}\eta}\left[z^{\mathcal{N}-1}\left(z^{\gamma}[\mathcal{P}(\eta)]^{\eta}\right)\right]-\overline{\alpha}[\mathcal{P}(\eta)]^{\overline{\mu}} \tag{26}$$

with $\eta=\zeta(t)r$ and k being a constant. To find a solution for the above equation is a hard task, leading us to cumbersome calculations. However, it is possible to obtain an implicit solution for $\mathcal{P}(\eta)$, which can be useful in developing a perturbation technique by using the procedure of iteration. This solution can be obtained to $\overline{\mu}=(2+\delta n+\theta)/\mathcal{N}+n+\nu$ and is given by

$$\mathcal{P}(\eta)=\exp_q^{n+1}\left[-\frac{1}{n+1}\left(\frac{k}{\mathcal{N}\delta^n\nu\mathcal{D}}\right)^{1/(n+1)}\int^{\eta}\mathrm{d}\overline{z}\,\overline{z}^{(1+\theta)(1+n)}\left(1+\frac{\mathcal{K}}{k}\mathcal{N}z^{\gamma}\mathcal{P}(\overline{z})^{\overline{\eta}-1} +\frac{\alpha\mathcal{N}}{k\overline{z}^{\mathcal{N}}\mathcal{P}(\overline{z})}\int^{\overline{z}}\mathrm{d}z'\,z'^{\mathcal{N}-1}[\mathcal{P}(z')]^{\overline{\mu}}\right)^{1/(n+1)}\right]. \tag{27}$$

Equation (27) recovers equation (11) for $\overline{\alpha}=0$ after performing a integration on $\tilde{z}$. We may use equation (27) to obtain the time behavior of the mth moments of even order related to this distribution for μ,n,ν and θ arbitraries by considering that the integrals involving the distribution are convergent. They are

$$\begin{aligned}\langle r^{2m}\rangle&=\left[\int\mathrm{d}r\,r^{\mathcal{N}-1}r^{2m}\rho(r,t)\right]\Big/\left[\int\mathrm{d}r\,r^{\mathcal{N}-1}\rho(r,t)\right]\\ \langle r^{2m}\rangle&=\zeta(t)^{-2m}\left[\int\mathrm{d}\eta\,\eta^{\mathcal{N}-1}\eta^{2m}\rho(\eta)\right]\Big/\left[\int\mathrm{d}\eta\,\eta^{\mathcal{N}-1}\rho(\eta)\right]\\ \langle r^{2m}\rangle&\propto\zeta(t)^{-2m}.\end{aligned} \tag{28}$$

By using the above result, we obtain $\langle r^2 \rangle \sim t^{2(n+\nu-\mu)/((\mu-1)(2+n+\theta))}$, with $2(n+\nu-\mu)/(2+n+\theta)$ less than, equal to and greater than one corresponding to sub-, normal and superdiffusion, respectively.

Let us address our discussion to the case $\mu' \neq 1$ and $\mu \neq 1$. In this manner, we extend the usual derivative operator, applied to the spatial variable, for the fractional case [34] in order to incorporate a noninteger index. For this case, we work out equation (1) in the absence of external forces and without an absorbent term:

$$\frac{\partial}{\partial t}\rho(r,t) = \frac{1}{r^{\mathcal{N}-1}}\frac{\partial}{\partial r}\left\{ r^{\mathcal{N}-1}\left[\mathcal{D}(t)r^{-\theta}\left|\frac{\partial^{\mu'}}{\partial r^{\mu'}}[\rho(r,t)]^{\delta}\right|^{n}\frac{\partial^{\mu}}{\partial r^{\mu}}[\rho(r,t)]^{\nu}\right]\right\} \tag{29}$$

which has as particular cases the situations analyzed in [40]–[44]. In order to obtain the solution for this equation, we employ the same procedure as above, which is based on equation (2) and leads us to ordinary differential equations. By substituting equation (2) in equation (29), and performing some calculations, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{ z^{\mathcal{N}-1}\left[z^{-\theta}\left|\frac{\mathrm{d}^{\mu'}}{\mathrm{d}z^{\mu'}}[\tilde{\rho}(z)]^{\delta}\right|^{n}\frac{\mathrm{d}^{\mu}}{\mathrm{d}z^{\mu}}[\tilde{\rho}(z)]^{\nu}\right]\right\} = \widetilde{k}\frac{\mathrm{d}}{\mathrm{d}z}[z^{\mathcal{N}}\tilde{\rho}(z)] \tag{30}$$

and

$$[\Phi(t)]^{\widetilde{\xi}-2}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = -\widetilde{k}\,\overline{\mathcal{D}}(t) \tag{31}$$

where $\widetilde{\xi} = 2+\mu+\mu' n+\theta+(\nu+\delta n-1)\mathcal{N}$, $\widetilde{k}$ is a constant and $z = r/\Phi(t)$, as before. Equation (31) has the same form of equation (6), consequently, the same solution. To find a solution for equation (30) is a hard task due to the presence of the fractional operators and nonlinearity in the diffusive term. For this reason, following [42], we propose the ansatz $\tilde{\rho}(z) = \widetilde{\mathcal{N}} z^{\widetilde{\alpha}}(1+bz)^{\widetilde{\beta}}$ as a solution, with the parameters $\widetilde{\alpha}$ and $\widetilde{\beta}$ to be determined in terms of the parameters μ, μ', θ, $\mathcal{N}$, ν and δ present in equation (30). In particular, after substituting this ansatz in equation (30) and performing some calculations, it is possible to show that

$$\begin{aligned}\widetilde{\alpha} &= \frac{1+\theta+\mu+n\mu'}{\delta n+\nu-1}\\ \widetilde{\beta} &= \frac{\mu+n\mu'}{\delta n+\nu-1}\\ \widetilde{\mathcal{N}} &= \left[\frac{\Gamma(\alpha\nu+1)}{\Gamma(\alpha\nu+1-\mu)}\left|\frac{\Gamma(\alpha\delta+1)}{\Gamma(\alpha\delta+1-\mu')}\right|^{n}\right]^{1/(1-(n\delta+\nu))}\end{aligned} \tag{32}$$

with ν, δ, μ' and μ subjected to the constraint $\delta(\mu'-1) = \nu(\mu-1)$. Thus, the solution for equation (30) can be written as

$$\widetilde{\rho}(z) = \widetilde{\mathcal{N}} z^{(1+\theta+\mu+n\mu')(\delta n+\nu-1)}(1+bz)^{(\mu+n\mu')/(\delta n+\nu-1)} \tag{33}$$

where b is a constant that should be taken as ± 1 according to the situation under analysis. It is important to stress that a choice for $b=-1$ implies a limited region for the solution, while for $b=1$ the solution covers the entire space. In this way, depending on the choice of the parameters μ, θ, ν, μ', n and δ the solution may present a compact ($b=-1$) or long tail behavior ($b=1$). We emphasize that the last case can be asymptotically related to the Lévy distributions and consequently with the distributions that emerge from the Tsallis formalism [52] which have a power law behavior. This connection is interesting enough to motivate a possible thermostatistical context different from the usual one.

3. Discussion and conclusion

In this work, equation (1), which can be used to investigate a broad range of situations, was analyzed considering several different scenarios, for example, time-dependent diffusion coefficient, the presence of external forces and reaction terms. For each case, analytical explicit or implicit solutions were obtained. These solutions obtained were expressed in terms of the q-exponential and q-logarithmic functions presents in the Tsallis formalism which suggests a different thermostatistical context for this equation. They may exhibit a short or long tail behavior depending on the choice of the parameters μ, μ', θ, δ, n, ν present in equation (1) where for long tail behavior a connection to Lévy distributions could be established, representing a superdiffusive process. Towards this direction, the analysis of the second moment represents an important tool, and the choice of the values of the parameters may lead to the model of a subdiffusive, normal or superdiffusive process. Finally, we hope that the results presented here can be useful to analyze situations related to anomalous diffusion.

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References

[1] Hilfer R, 2000 *Applications of Fractional Calculus in Physics* (Singapore: World Scientific)
[2] Metzler R and Klafter J, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, 2000 *Phys. Rep.* **339** 1
[3] West B J, Bologna M and Grigolini P, 2002 *Physics of Fractal Operators* (New York: Springer)
[4] Klafter J, Shlesinger M F and Zumofen G, *Beyond Brownian motion*, 1996 *Phys. Today* **49** 33
[5] Pekalski A and Sznajd-Wero K, 1999 *Anomalous Diffusion: From Basics to Applications* (*Springer Lecture Notes in Physics*) (Telos: Springer)
[6] Bouchaud JP and Georges A, *Anomalous diffusion in disordered media—statistical mechanisms, models and physical applications*, 1990 *Phys. Rep.* **195** 127
[7] Campos D and Mendez V, *Description of diffusive and propagative behavior on fractals*, 2004 *Phys. Rev.* E **69** 031115
[8] O'Shaughnessy B and Procaccia I, *Analytical solutions for diffusion on fractal objects*, 1985 *Phys. Rev. Lett.* **54** 455
[9] Crothers D S F, Holland D, Kalmykov Y P and Coffey W T, *The role of Mittag–Leffler functions in anomalous relaxation*, 2002 *J. Mol. Liq.* **114** 27
[10] Iomin A, *Toy model of fractional transport of cancer cells due to self-entrapping*, 2006 *Phys. Rev.* E **73** 061918
[11] Muskat M, 1937 *The Flow of Homogeneous Fluid Through Porous Media* (New York: McGraw-Hill)
[12] Plerou V, Gopikrishnan P, Amaral L A N, Gabaix X and Stanley H E, *Economic fluctuations and anomalous diffusion*, 2000 *Phys. Rev.* E **62** R3023
[13] Peng C K, Mietus J, Hausdorff J M, Havlin S, Stanley H E and Goldberger A L, *Long-range anticorrelations and non-Gaussian behavior of the heartbeat*, 1990 *Phys. Rev. Lett.* **65** 2201
[14] Sher H and Montroll E W, *Anomalous transit-time dispersion in amorphous solids*, 1975 *Phys. Rev.* B **12** 2455

[15] Ott A, Bouchaud J P, Langevin D and Urbach W, *Anomalous diffusion in living polymers: a genuine Levy flight?*, 1998 *Phys. Rev. Lett.* **80** 5015
[16] Malacarne L C, Mendes R S, Pedron I T and Lenzi E K, *N-dimensional nonlinear Fokker–Planck equation with time-dependent coefficients*, 2002 *Phys. Rev.* E **65** 52101
[17] Shlesinger M F, Zaslavsky G M and Frisch U, 1994 *Lévy Flights and Related Topics in Physics* (Berlin: Springer)
[18] Balescu R, *V-Langevin equations, continuous time random walks and fractional diffusion*, 2007 *Chaos Solitons Fractals* **34** 62
[19] Gardiner C W, 1996 *Handbook of Stochastic Methods: For Physics, Chemistry and the Natural Sciences* (*Springer Series in Synergetics*) (New York: Springer)
[20] Metzler R, Barkai E and Klafter J, *Anomalous transport in disordered systems under the influence of external fields*, 1999 *Physica* A **266** 343
[21] Plotkin S S and Wolynes P G, *Non-Markovian configurational diffusion and reaction coordinates for protein folding*, 1998 *Phys. Rev. Lett.* **94** 170602
[22] Kosztoowicz T, Dworecki K and Mrówczy S S T, *How to measure subdiffusion parameters?*, 2005 *Phys. Rev.* E **72** 061918
[23] Khan Z S and Morris S W, *Subdiffusive axial transport of granular materials in a long drum mixer*, 2005 *Phys. Rev. Lett.* **94** 048002
[24] Lua R C and Grosberg A Y, *First passage times and asymmetry of DNA translocation*, 2005 *Phys. Rev.* E **72** 061918
[25] Muskat M, 1937 *The Flow of Homogeneous Fluids Through Porous Media* (New York: McGraw-Hill)
[26] Polubarinova-Kochina P Y, 1962 *Theory of Ground Water Movement* (Princeton, NJ: Princeton University Press)
[27] Buckmaster J, *Viscous sheets advancing over dry beds*, 1977 *J. Fluid Mech.* **81** 735
[28] Spohn H, *Surface dynamics below the roughening transition*, 1993 *J. Physique* I **3** 69
[29] Richardson L F, *Atmospheric diffusion shown on a distance–neighbour graph*, 1926 *Proc. R. Soc.* A **110** 709
[30] Frisch U, 1995 *Turbulence: the Legacy of A N Kolmogorov* (New York: Cambridge University Press)
[31] Shaughnessy B and Procaccia I, *Diffusion on fractals objects*, 1985 *Phys. Rev. Lett.* **54** 455
[32] Vedenov, 1967 *Rev. Plasma Phys.* **3** 229
[33] Su N, *Generalisation of various hydrological and environmental transport models using Fokker–Planck equation*, 2004 *Environ. Mod. Softw.* **19** 345
[34] Podlubny I, 1999 *Fractional Differential Equations* (San Diego, CA: Academic)
[35] Stephenson J, *Some non-linear diffusion equations and fractal diffusion*, 1995 *Physica* A **222** 234
[36] Sokolov I M, Schmidt M G W and Sagués F, *Reaction-subdiffusion equations*, 2006 *Phys. Rev.* E **73** 031102
[37] Sagués F, Shkilev V P and Sokolov I M, *Reaction-subdiffusion equations for the $A \rightleftharpoons B$ reaction*, 2008 *Phys. Rev.* E **77** 032102
[38] Langlands T A M, Henry B I and Wearne S L, *Turing pattern formation with fractional diffusion and fractional reactions*, 2007 *J. Phys.: Condens. Matter* **19** 065115
[39] Burgers J M, 1974 *The Nonlinear Diffusion Equation* (Dordrecht: Reidel)
[40] Lenzi E K, Mendes G A, Mendes R S, Silva L R and Lucena L S, *Exact solutions to nonlinear nonautonomous space-fractional diffusion equations with absorption*, 2003 *Phys. Rev.* E **67** 051109
[41] Bologna M, Tsallis C and Grigolini P, *Anomalous diffusion associated with nonlinear fractional derivative Fokker–Planck-like equation: exact time-dependent solutions*, 2000 *Phys. Rev.* E **62** 2213
[42] Tsallis C and Lenzi E K, *Anomalous diffusion: nonlinear fractional Fokker–Planck equation*, 2002 *Chem. Phys.* **284** 341
[43] Lenzi E K, Malacarne L C, Mendes R S and Pedron I T, *Anomalous diffusion, nonlinear fractional Fokker–Planck equation and solutions*, 2003 *Physica* A **319** 245
[44] Lenzi E K, Mendes R S, Fa K S, Moraes L S, da Silva L R and Lucena L S, *Nonlinear fractional diffusion equation: exact results*, 2005 *J. Math. Phys.* **46** 083506
[45] Vlad M O, *Fractional diffusion equation on fractals—self-similar stationary solutions in a force-field derived from a logarithmic potential*, 1994 *Chaos Solitons Fractals* **4** 191
[46] Havlin S and Ben-Avraham D, *Diffusion in disordered media*, 1987 *Adv. Phys.* **36** 695
[47] Alemany P A, Zanette D H and Wio H S, *Time-dependent reactivity for diffusion-controlled annihilation and coagulation in two dimensions*, 1994 *Phys. Rev.* E **50** 3646
[48] Lee B P, *Renormalization-group calculation for the reaction $kA \to \emptyset$*, 1994 *J. Phys. A: Math. Gen.* **27** 2633
[49] Crank J, 1956 *The Mathematics of Diffusion* (London: Oxford University Press)
[50] Carslaw H S and Jaeger J C, 1959 *Conduction of Heat in Solids* (London: Oxford University Press)

J. Stat. Mech. (2009) P02048

[51] Pascal J P and Pascal H, *On some diffusive waves in nonlinear heat-conduction*, 1993 *Int. J. Non-Linear Mech.* **28** 641

[52] Abe S and Okamoto Y, *Nonextensive statistical mechanics and its applications*, 2001 *Springer Lecture Notes in Physics* (Heidelberg: Springer)

[53] Frank T D, *A Langevin approach for the microscopic dynamics of nonlinear Fokker–Planck equations*, 2001 *Physica* A **301** 52