q-distributions in complex systems: a brief review

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The nonextensive statistical mechanics proposed by Tsallis is today an intense and growing research field. Probability distributions which emerges from the nonextensive formalism (q-distributions) have been applied to an impressive variety of problems. In particular, the role of q-distributions in the interdisciplinary field of complex systems has been expanding. Here, we make a brief review of q-exponential, q-Gaussian and q-Weibull distributions focusing some of their basic properties and recent applications. The richness of systems analyzed may indicate future directions in this field.

Keywords: q-exponential, q-Gaussian, q-Weibull, Nonextensive statistics

1. INTRODUCTION

Common characteristics of complex systems include long-range correlations, multifractality and non-Gaussian distributions with asymptotic power law behavior. Typically, such systems are not well described by approaches based on the usual statistical mechanics. In this scenario, a new formalism capable of providing a better description of complex systems is welcome. This is the case of the generalized (nonextensive) statistical mechanics proposed by Tsallis - nowadays, an intense and growing research field[1–4].

Concepts related with nonextensive statistical mechanics have found applications in a variety of disciplines including physics, chemistry, biology, mathematics, geography, economics, medicine, informatics, linguistics among others[5–7]. Probability distributions which emerge from the nonextensive formalism - also called q-distributions - have been applied to an impressive variety of problems in diverse research areas including the interdisciplinary field of complex systems.

In the present work we focus on q-exponential, q-Gaussian and q-Weibull distributions. We summarized some of their basic properties and provide useful references of recent applications. The richness of systems analyzed may indicate future directions in this research line.

2. q-EXPONENTIAL DISTRIBUTION

The q-exponential distribution is given by the probability density function (pdf)

$$p_{qe}(x) = p_0 \left[1-(1-q)\,\frac{x}{x_0}\right]^{1/(1-q)} \tag{1}$$

for $1-(1-q)x/x_0 \geq 0$. If $p_0 = (2-q)/x_0$, eq. (1) is normalized.

In the limit $q \to 1$, eq. (1) recovers the usual exponential distribution in the same way in which the q-exponential function, defined as $e_q^{-x} \equiv [1-(1-q)x]^{1/(1-q)}$, recovers exponential function in the limit $q \to 1$ ($e_1^{-x} \equiv e^{-x}$). If $q < 1$, eq. (1) has a finite value for any finite real value of x since, by definition, $p_{qe}(x) = 0$ for $1-(1-q)x/a < 0$. If $q > 1$, eq. (1) exhibits power law asymptotic behavior,

$$p_{qe}(x) \sim x^{-1/(q-1)}. \tag{2}$$

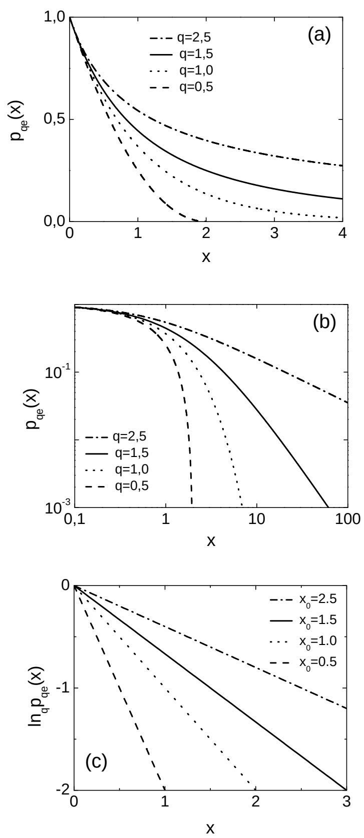

FIG. 1: **q-exponential distribution**. a) Plot of $p_{qe}(x)$ versus x, with $p_0 = x_0 = 1$ and typical values of q. b) Log-log plot of the curves in a). c) $\ln_q p_{qe}(x)$ versus x for $p_0 = 1$ and typical values of x_0.

Note also that $p_{qe}(x) \simeq 1+x$ for small x, independently of the q value. Figures 1a and 1b show $p_{qe}(x)$ versus x for typical values of q.

The q-exponential distribution, for $q > 1$, corresponds to the Zipf-Mandelbrot law[8] and a Burr-type distribution[9]. In this sense, the q-exponential is a generalization of these distributions for $q < 1$. Thus, by choosing suitable values for q, q-exponentials may be used to represent both short and long tailed distributions. This feature also holds for the other q-distributions.

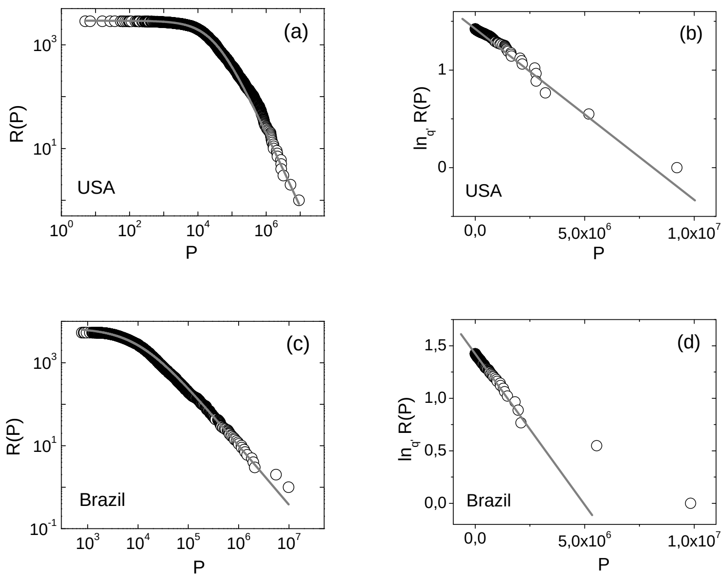

FIG. 2: **Population of cities.** a) Empirical cdf $R(P)$, where P is the population of USA cities. The solid line is a q-exponential, given by eq. (3), with $q' = 1.7$ ($q \simeq 1.4$), $x'_0 = 21,250$ and $c' = 2,919$. b) $\ln_{q'} R(P)$ versus P, with $q' = 1.7$, for the same data shown in (a). The solid line is a linear fit to the data. c) Empirical cdf $R(P)$, where P is the population of Brazilian cities. The solid line is a q-exponential, given by eq. (3), with $q' = 1.7$ ($q \simeq 1.4$), $x'_0 = 7,073$ and $c' = 6,968$. d) $\ln_{q'} R(P)$ versus P, with $q' = 1.7$, for the same data shown in (c). The solid line is a linear fit to the data.

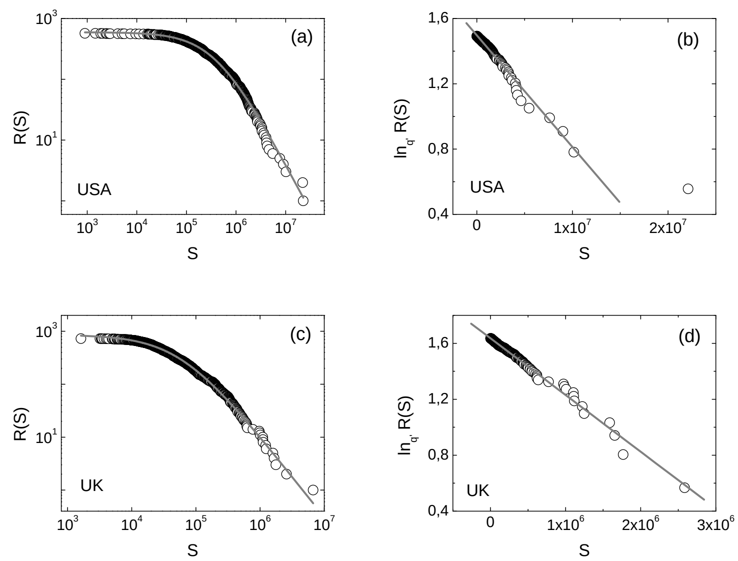

FIG. 3: **Circulation of magazines.** a) Empirical cdf $R(S)$, where S is the circulation of 570 USA magazines in 2004. The solid line is a q-exponential, given by eq. (3), with $q' = 1.65$ ($q \simeq 1.4$), $x'_0 = 255,204$ and $c' = 594$. b) $\ln_{q'} R(S)$ versus S, with $q' = 1.65$, for the same data shown in (a). The solid line is a linear fit to the data. c) Empirical cdf $R(S)$, where S is the circulation of 727 UK magazines in 2005. The solid line is a q-exponential, given by eq. (3), with $q' = 1.65$ ($q \simeq 1.4$), $x'_0 = 37,493$ and $c' = 860$. b) $\ln_{q'} R(S)$ versus S, with $q' = 1.65$, for the same data shown in (c). The solid line is a linear fit to the data.

The cumulative distribution function (cdf) associated to eq. (1) is given by

$$R_{qe}(x) = \int_x^\infty p_{qe}(y)\,dy = p'_0 \left[1-(1-q')\frac{x}{x'_0}\right]^{1/(1-q')}, \qquad (3)$$

defined for $q < 2$, with $q' = 1/(2-q)$, $x'_0 = x_0/(2-q)$ and $p'_0 = p_0 x_0/(2-q)$. Observe that $R_{qe}(x)$ and $p_{qe}(x)$ exhibit the same mathematical form.

It is possible to visualize q-exponential distributions as straight lines in graphs with appropriate scales. Applying the q-logarithm function, defined as $\ln_q x \equiv [x^{(1-q)} - 1]/(1-q)$, with $\ln_1 x \equiv \ln(x)$, in both sides of eq. (1), we have

$$\ln_q p_{qe}(x) = \ln_q p_0 - [1+(1-q)\ln_q p_0]\frac{x}{x_0}. \qquad (4)$$

A similar result holds for $R_{qe}(x)$. Figure 1c shows $\ln_q p_{qe}(x)$ versus x for typical values of x_0.

The q-exponential function given by eq. (1) has been employed in a growing number of theoretical and empirical works on a large variety of themes. Examples include scale-free networks[10–14], dynamical systems[15–27], algebraic structures[28–31] among other topics in statistical physcics[32–36].

As specific examples of q-exponential distributions in complex systems, let us consider results on population of cities[37] and circulation of magazines[38]. Figure 2 shows the cumulative distribution of the population of cities in the USA and Brazil. Figure 3 shows the cumulative distribution of circulation of magazines in the USA and UK. In both cases - population of cities and circulation of magazines - the empirical data are consistent with a q-exponential distribution, with $q \simeq 1.4$.

q-exponential distributions have also been applied in the empirical study of stock markets[39–42], DNA sequences[43], family names[44], human behavior[45–47], geomagnetic records[48, 49], train delays[50], reaction kinetics[51], air networks[52], hydrological phenomena[53], fossil register[54], basketball[55], earthquakes[56–58], world track records[59], voting processes[60], internet[61], individual success[62], citations of scientific papers[63, 64], football[65], linguistics[66, 67] and solar neutrinos[68, 69].

3. q-GAUSSIAN DISTRIBUTION

The q-Gaussian distribution is specified by the pdf

$$p_{qg}(x) = p_0 \left[1-(1-q)\left(\frac{x}{x_0}\right)^2\right]^{1/(1-q)}, \quad (5)$$

for $1-(1-q)(x/x_0)^2 \geq 0$ and $p_{qg}(x) = 0$ otherwise. It is normalized if $p_0 = (2/x_0)\sqrt{(q-1)/\pi}\,\Gamma[1/(q-1)]/\Gamma[1/(q-1)-1/2]$. In addition, eq. (5) presents unit variance if $x_0^2 = 5-3q$, with $q < 5/3$.

In the limit $q \to 1$, eq. (5) recovers the usual Gaussian distribution, so $q \neq 1$ indicates a departure from Gaussian statistics. For $q > 1$, the tails of q-Gaussian decrease as power laws,

$$p_{qg}(|x|) \sim |x|^{-2/(q-1)}. \quad (6)$$

Figures 4a and 4b show $p_{qg}(x)$ for typical values of q.

Applying the q-logarithm function in both sides of eq. (5), we have

$$\ln_q p_{qg}(x) = \ln_q p_0 - [1+(1-q)\ln_q p_0]\left(\frac{x}{x_0}\right)^2. \quad (7)$$

Figure 4c shows $\ln_q p_{qg}(x)$ versus x^2 for typical values of x_0.

Recent works have been focused on the study of mathematical properties of q-Gaussian functions[70–78], including methods for generating random numbers which follow q-Gaussian distributions[79, 80]. q-Gaussians have been employed in the study of a wide range of themes including probabilistic models[81, 82], stellar plasmas[83], porous-medium equation[84], Bose-condensed gases[85–87],

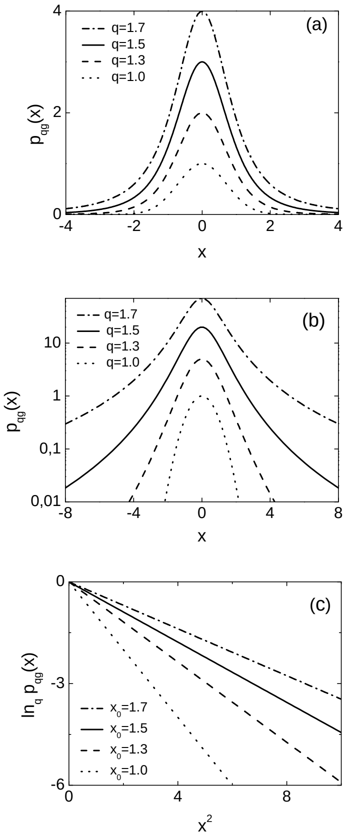

FIG. 4: **q-Gaussian distribution**. a) Plot of $p_{qg}(x)$ versus x, with $p_0 = x_0 = 1$, for typical values of q. Some curves were vertically shifted for a better visualization. b) The same curves shown in a), but for mono-log scale. Some curves were also shifted . c) $\ln_q p_{qg}(x)$ versus x^2 for $p_0 = 1$ and typical values of x_0.

dynamical systems[88–90], polymeric networks[91], small-world networks[92], fingering processes[93], processes with stochastic volatility[94, 95] and nonlinear diffusion[96, 97].

In order to illustrate a recent application of q-Gaussian distributions in complex systems, we mention here results on the dynamics of earthquakes[98]. Figure 5 shows the distribution of energy differences between successive earthquakes at the San Andreas Fault. The empirical data is consistent with a q-Gaussian distribution, with $q = 1.75$.

Other recent applications of q-Gaussian distribution include stock markets[99–107], DNA molecules[108], the solar wind[109–111], galaxies[112], optical lattices[113], cellular aggregates[114] and the atmosphere[115].

4. q-WEIBULL DISTRIBUTION

The q-Weibull distribution is given by the pdf

$$p_{qw}(x) = p_0 \frac{rx^{r-1}}{x_0^r}\left[1-(1-q)\left(\frac{x}{x_0}\right)^r\right]^{1/(1-q)}, \quad (8)$$

for $1-(1-q)(x/x_0)^r \geq 0$ and $p_{qw}(x) = 0$ otherwise. Eq. (8) is normalized if $p_0 = 2-q$.

In the limits $q \to 1$, $r \to 1$, and $q \to 1$, $r \to 1$, eq. (8) recovers Weibull, q-exponential and exponential distributions, re-